Lecture 19

Abelian additive difference equations

(19.0) Euler's Gamma function.

Definition. Let $\gamma = \lim_{m\to\infty} \left(1+\frac{1}{2}+\dots+\frac{1}{m} - \log(m)\right)$: Euler-Mascheroni constant

Define $\phi(z) = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$.

Lemma. $z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$ represents an analytic function of z with simple zeros at $z \in -\mathbb{N}$.

Proof. Let us consider the compact set $|z| \le \frac{N}{2}$ (N: fixed positive integer)

Then for $n > N$:

$$\left|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right| = \left| -\frac{1}{2}\frac{z^2}{n^2} + \frac{1}{3}\frac{z^3}{n^3} + \cdots \right|$$

$$\le \frac{|z|^2}{n^2}\left(1 + \left|\frac{z}{n}\right| + \left|\frac{z^2}{n^2}\right| + \cdots\right) \le \frac{1}{4}\frac{N^2}{n^2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)$$

$$\le \frac{N^2}{2n^2}$$

Since $\sum_{n=N+1}^{\infty} N^2/2n^2$ converges, it follows that, when $|z| \le N/2$

$\sum_{n=N+1}^{\infty} \left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right)$ is absolutely and uniformly convergent series of analytic functions. This proves the lemma $\square$

$$\Gamma(z) := \frac{1}{\phi(z)} = z^{-1} e^{-\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \quad \text{(Weierstrass)}$$

is meromorphic function with only simple poles at $-\mathbb{N}$ (and no zeroes).

(19.1) Euler's formula: $\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{z} \left(1+\frac{z}{n}\right)^{-1}$

Proof: $$\frac{1}{\Gamma(z)} = z \cdot \exp\left(\lim_{m\to\infty} 1+\frac{1}{2}+\dots+\frac{1}{m}-\log m\right) \left(\lim_{m\to\infty} \prod_{n=1}^{m}\left(1+\frac{z}{n}\right)e^{-z/n}\right)$$

$$= z \lim_{m\to\infty}\left[m^{-z}\prod_{n=1}^{m}\left(1+\frac{z}{n}\right)\right] = z\lim_{m\to\infty}\left(\prod_{n=1}^{m-1}\left(1+\frac{1}{n}\right)^{-z}\prod_{n=1}^{m}\left(1+\frac{z}{n}\right)\right) \qquad \square$$

Cor: $\Gamma(z+1) = z\,\Gamma(z)$. and $\Gamma(1)=1$. ($\Rightarrow \Gamma(n)=(n-1)!$ $\forall\, n\geq 1$)

Proof: $$\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{z}{z+1}\lim_{m\to\infty}\prod_{n=1}^{m}\frac{\left(1+\frac{1}{n}\right)(z+n)}{z+n+1}$$

$$= z\lim_{m\to\infty}\frac{m+1}{z+m+1} = z \qquad \square$$

(19.2) $$\Gamma(z)\,\Gamma(-z) = -\frac{1}{z^2}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{-\pi}{z\sin(\pi z)}$$

$\left(\frac{\sin z}{z} = \prod_{n=1}^{\infty} 1-\frac{z^2}{(n\pi)^2}\right)$. Hence

$$\boxed{\Gamma(1+z)\,\Gamma(1-z) = \frac{2\pi i z}{e^{\pi i z}-e^{-\pi i z}}}$$

(19.3) Let $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ (logarithmic derivative of Γ).

$$\psi(z+1) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{z\Gamma'(z)}{z\Gamma(z)} + \frac{\Gamma(z)}{z\Gamma(z)} = \psi(z)+\frac{1}{z}.$$

(19.4) Stirling Series : $\Gamma(z)$ is asymptotic to

$$e^{-z} \, z^{z-\frac{1}{2}} \, (2\pi)^{\frac{1}{2}} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} \cdots \right)$$

as $z \to \infty$ in $|\arg z| \leq \pi - \Delta$ (Δ : fixed positive real)

Or $\log\Gamma(z) - \left(z - \frac{1}{2}\right)\log z + z - \frac{1}{2}\log(2\pi)$ is asymptotic to $\displaystyle\sum_{r=1}^{\infty} \frac{(-1)^{r-1} B_r}{2r(2r-1)\, z^{2r-1}}$

(recall $f(z)$ is asymptotic to $\displaystyle\sum_{n\geq 0} a_n z^{-n}$ if $\forall\, n$

$$\lim_{|z|\to\infty} \left| z^n \left(f(z) - \sum_{k=0}^{n} a_k z^{-k} \right) \right| = 0$$; for z lying in a domain (unbounded))

$\{B_n\}_{n\geq\varphi}$ are Bernoulli's numbers defined by

$$\frac{1}{2} x \cot\left(\frac{x}{2}\right) = 1 - \sum_{n=1}^{\infty} B_n \frac{x^{2n}}{(2n)!}$$ (near $x=0$) .

(19.5) Additive difference equations

V: f.d. $\mathbb{C}$-vector space $A : \mathbb{C} \to \mathrm{End}V$ rational fn. regular at ∞

- $A(\infty) = 1$. Consider the Taylor series expansion near $u = \infty$.

$$A(u) = 1 + A_0 u^{-1} + A_1 u^{-2} + \cdots$$

- Non-resonance : eigenvalues of $\mathrm{ad}\, A_0$ acting on the subalgebra $\mathcal{A} \subset \mathrm{End}V$ generated by $\{A_k\}_{k\geq 0}$ do not lie in $\mathbb{Z}^{\times}$.
(e.g. if $[A_0, A(u)] = 0 \;\; \forall u$; or if eigenvalues of A_0 do not differ by $\mathbb{Z}^{\times}$)

Consider the following equation for a mero. fn. $\phi : \mathbb{C} \to \mathrm{End}V$

$$\boxed{\phi(u+1) = A(u)\,\phi(u).} \qquad (\star)$$

Similar to the theory of q- difference equations, one can construct formal solution $\Upsilon(u) = \left(1 + y_0 u^{-1} + y_1 u^{-2} + \cdots\right) u^{A_0}$ as follows:

$$Y(u+1) = A(u)\, Y(u)$$

$$\iff \left(1+\sum_{r\geq 0} y_r (u+1)^{-r-1}\right) = \left(1+\sum_{r\geq 0} A_r u^{-r-1}\right)\left(1+\sum_{r\geq 0} y_r u^{-r-1}\right)(1+u^{-1})^{A_0}$$

Write $(1+u^{-1})^{A_0} = \sum_{r\geq 0} (-1)^r \binom{A_0}{r} u^{-r}$ where $\binom{A_0}{r} = \dfrac{A_0(A_0+1)\dots(A_0+r-1)}{r!}$

Gives a recursive method to compute y_n.

However $1+\sum_{r\geq 0} y_r u^{-r-1}$ is not necessarily convergent near ∞.

Theorem (Birkhoff 1911): There exist 2 (uniqueness) mero. soln.s $\phi^\pm : \mathbb{C} \to \mathrm{End}\, V$ s.t.

Fundamental Solutions
(1) $\phi^\pm$ are hol. & inv. for $\pm \mathrm{Re}(u) \gg 0$
(2) $\phi^\pm(u) \sim H^\pm(u)\,(\pm u)^{A_0}$ in any right/left half plane.

Moreover: (3) Poles of $\phi^+ \subset$ (Poles of $A(u)^{-1}$) $- \mathbb{N}$
Poles of $\phi^- \subset$ (Poles of $A(u)$) $+ \mathbb{N}^\times$

(4) $H^\pm(u) = 1+\sum_{r\geq 0} y_r u^{-r-1}$

Define $\boxed{S(u) := \phi^+(u)^{-1}\phi^-(u)}$. Then as a function of $z = e^{2\pi i u}$, S is rational, regular at $z=0$ and ∞ and:

$$S(\infty) = e^{\pi i A_0} = S(0)^{-1}$$

Connection (or monodromy) matrix.

Coefficient Matrices	$\rightsquigarrow$	Fundamental Solutions	$\rightsquigarrow$	Monodromy matrix
$A(u)$		$\phi^\pm(u)$		$S(z)$
				$z = e^{2\pi i u}$
Rational				Trigonometric

Now consider the inverse problem:

Given : $A_0 \in \mathrm{End}(V)$ $\quad S : \mathbb{C} \to \mathrm{End}\,V$ rat'l fn. of $e^{2\pi i u}$ s.t.

$$S(z=0)^{-1} = S(z=\infty) = e^{\pi i A_0}$$

Riemann-Hilbert factorization problem: find 2 mero. fns $\phi^{\pm} : \mathbb{C} \to \mathrm{End}\,V$ s.t.

- $\phi^{\pm}$ are hol. & inv for $\pm \mathrm{Re}(u) \gg 0$
- $\phi^{\pm}$ have asymptotic expansion $\left(1 + \sum_{r \geq 0} h_r^{\pm} u^{-r-1}\right)(\pm u)^{A_0}$ (in any right/left half plane)
- $\phi^{+}(u)^{-1} \phi^{-}(u) = S(u)$.

Set $A(u) = \phi^{+}(u+1)\,\phi^{+}(u)^{-1} = \phi^{-}(u+1)\,\phi^{-}(u)^{-1}$

Prop. (Uniqueness of factorization)

Let $A(u)$ and $A'(u)$ be two coefficient matrices (with same A_0) with same monodromy. Let

P = poles of $A(u)$ $\qquad$ P' = poles of $A'(u)$

Z = poles of $A(u)^{-1}$ $\qquad$ Z' = poles of $A'(u)^{-1}$

If all pairs (Z, P), (Z', P'), (Z, Z'), (P, P') are <u>non-congruent</u>*

then $A(u) = A'(u)$

(we say (X, Y) is non-congruent pair for $X, Y \overset{\text{finite}}{\subset} \mathbb{C}$ if $x - y \notin \mathbb{Z}^{\times}$ $\forall\, x \in X, y \in Y$)

Remark : if $\phi_1^{\pm}$ and $\phi_2^{\pm}$ are two solutions of factorization problem

then $G(u) = \phi_2^{+}(u)\,\phi_1^{+}(u)^{-1} = \phi_2^{-}(u)\,\phi_1^{-}(u)^{-1}$ becomes a rat'l fn. of u

$G(\infty) = 1$.

Corresponding coeff. matrices are then related by

$$A_{\phi_2}(u) = G(u+1)\,A_{\phi_1}(u)\,G(u)^{-1}$$

(called isomonodromy transformation)

(19.6) Some proofs in generality of (19.5)

- Uniqueness part of the theorem. Let ϕ_1^+ and ϕ_2^+ be two mero. solns. of (★) satisfying (1) and (2) of Theorem (19.5).

$$C(u) := \phi_1^+(u)^{-1}\,\phi_2^+(u) \quad \text{is 1-periodic and hol. for } \operatorname{Re}(u) \gg 0.$$

Hence $C(u)$ is holomorphic on the entire complex plane.

Claim: As a function of $z = e^{2\pi i u}$ C has removable singularities at $z = 0, \infty$.

(Indeed, write $C(u) = u^{-A_0}\,\underset{\downarrow \atop 1}{\left(\phi_1^+(u)\,u^{-A_0}\right)^{-1}}\,\underset{\downarrow \atop 1}{\left(\phi_2^+(u)\,u^{-A_0}\right)}\,u^{A_0}$ as $\operatorname{Im}(u) \to \pm\infty$

1st and 4th terms only grow like a polynomial, hence

$$\lim_{z\to 0} z\,C(z) = 0 = \lim_{z\to\infty} z^{-1}C(z) \;)$$

Also $\boxed{u^{A_0}\,C\,u^{-A_0} \sim 1 + C^{\circ}u^{-1} + \dots}$ (a) from above.

Hence $C(u) = C$ is a constant matrix.

Let $A_0 = S_0 + N_0$ be Jordan dec. of A_0, and $C = \sum_{\lambda\in\mathbb{C}} C_\lambda$ be dec. of C into eigenvalues of $\operatorname{ad}(S_0)$.

$$\operatorname{Ad}(u^{A_0})\,C_\lambda = u^{\lambda}\sum_{k\geq 0}(\log u)^k\,\frac{(\operatorname{ad}N_0)^k}{k!}\,C_\lambda \qquad (b)$$

Comparing (a) and (b): $C_\lambda = 0$ for $\lambda \notin \mathbb{Z}_{\leq 0}$; C_λ is eigenvector for $\operatorname{ad}A_0$; $C_0 = 1$. Non-resonance $\Rightarrow$ $C_\lambda = 0$ for $\lambda\in\mathbb{Z}_{<0}$. Hence $C = 1$.

- ~~Rationality of~~ $\cancel{S(z) = \phi^+(u)^{-1}\phi^-(u)\big|_{z=e^{2\pi i u}}}$

- (3) and (4) of Theorem (19.5):

(4) is clear from uniqueness of formal solution $\left(1+\sum_{r\geq 0} y_r u^{-r-1}\right)u^{A_0}$.

(3): $$\phi^+(u) = A(u)^{-1}\cdots A(u+n-1)^{-1}\,\underbrace{\phi^+(u+n)}_{\text{hol. for large } n}$$

$\Rightarrow$ Poles of $\phi^+ \subset \left(\text{Poles of } A(u)^{-1}\right) - \mathbb{N}$

- Rationality of $S(z)$ and its values at 0 & ∞:

$$S(z) = \phi^+(u)^{-1}\,\phi^-(u)\Big|_{z=e^{2\pi i u}}$$ has only finitely many poles in z-plane

$$S(u) = u^{A_0}\left(\phi^+(u)\,u^{A_0}\right)^{-1}\left(\phi^-(u)(-u)^{A_0}\right)(-u)^{-A_0}$$
$$\sim u^{A_0}(-u)^{-A_0} = e^{\pm\pi i A_0} \quad \left(\begin{array}{ll} + & \text{if } \operatorname{Im} u>0 \\ - & \text{if } \operatorname{Im} u<0\end{array}\right)$$

- Uniqueness of R-H factorization prop (19.5):

$$A'(u) = G(u+1)\,A(u)\,G(u)^{-1}$$

$\Rightarrow$ $G(u+1) = A'(u)\,G(u)\,A(u)^{-1}$. Hence $\forall\, n \in \mathbb{N}^\times$

$$G(u) = A'(u-1)\cdots A'(u-n)\,G(u-n)\,A(u-n)^{-1}\cdots A(u-1)^{-1}.$$

Since G is regular for $\operatorname{Re}(u) \ll 0$, we get that poles of $G \subset (\mathcal{P}' \cup \mathcal{Z}) + \mathbb{N}^\times$

Illy $G(u) = A'(u)^{-1}\cdots A'(u+n-1)^{-1}\,G(u+n)\,A(u+n-1)\cdots A(u)$

$\Rightarrow$ poles of $G(u) \subset (\mathcal{Z}' \cup \mathcal{P}) - \mathbb{N}$. Non-congruence assumption $\Rightarrow$

G has no poles, hence $G = 1$.

(19.7) Existence in the restricted case: namely <u>abelian case</u>.

Also I am going to assume A_0 is semisimple.

As usual abelian case splits into semisimple and unipotent cases

Semisimple case:

$$A(u) = \prod_{j\in J}\frac{u-a_j}{u-b_j} \,. \qquad \phi^+(u) = \prod \frac{\Gamma(u-a_j)}{\Gamma(u-b_j)}$$

$$\phi^-(u) = \prod\frac{\Gamma(1-u+b_j)}{\Gamma(1-u+a_j)}$$

$$S(u) = \prod \frac{\Gamma(u-b_j)\,\Gamma(1-u+b_j)}{\Gamma(u-a_j)\,\Gamma(1-u+a_j)}$$

$$= \prod \frac{e^{\pi i(u-a_j)} - e^{-\pi i(u-a_j)}}{e^{\pi i(u-b_j)} - e^{-\pi i(u-b_j)}} = e^{\pi i A_0} \prod_j \frac{z-\alpha_j}{z-\beta_j}$$

where $\alpha_j = e^{2\pi i a_j}$ $\quad \beta_j = e^{2\pi i b_j}$ $\quad z = e^{2\pi i u}$ $\quad A_0 = \sum b_j - a_j$

Conversely if S and A_0 are given, for every choice of $\{a_j, b_j\}$ s.t. $\sum b_j - a_j = A_0$, we have a solution of the factorization problem.
(unique if additionally non-congruence on $\{a_j, b_j\}$ is imposed).

Asymptotics of $\phi^{\pm}$:

$$\phi^+ \sim e^{\sum b_j - a_j} \prod_j \frac{(u-a_j)^{u-a_j+\frac{1}{2}}}{(u-b_j)^{u-b_j-\frac{1}{2}}} \sim u^{A_0}\left(1+O(u^{-1})\right)$$

Use the fact that $\dfrac{\Gamma(u-\lambda)}{\Gamma(u)} \sim u^{-\lambda}\left(1+O(u^{-1})\right)$

Unipotent case can be written additively $\quad (A(u) = 1 + N(u) : N$ is nilpotent$)$

$$\varphi(u+1) - \varphi(u) = a(u) \qquad a(u) = \log A(u) = \sum_{k\geq 1} (-1)^{k-1} \frac{N(u)^k}{k}$$

is still a rat'l fn. vanishing at $u = \infty$.

Writing partial fractions of each matrix entry of $a(u)$ reduces to

$$\varphi(u+1) - \varphi(u) = \frac{1}{(u-a)^{n+1}} \qquad (n \geq 0)$$

Let $\psi^+(u) = \dfrac{\Gamma'(u)}{\Gamma(u)}$ $\quad \psi^-(u) = \dfrac{\Gamma'(1-u)}{\Gamma(1-u)}$. Then

$$\psi^\pm(u+1) - \psi^\pm(u) = \frac{1}{u} \rightsquigarrow \frac{(-\partial_u)^r}{r!}\left(\psi^\pm(u+1) - \psi^\pm(u)\right) = \frac{1}{u^{r+1}}.$$

$$\Rightarrow \quad \varphi^\pm_{a,n} := \frac{(-\partial_u)^n}{n!}\,\psi^\pm(u-a)$$ are solutions for

$$\varphi(u+1) - \varphi(u) = \frac{1}{(u-a)^{n+1}}.$$

Also $$\psi^-(u) - \psi^+(u) = \pi\cot(\pi u) = \pi i\left(\frac{e^{2\pi i u}+1}{e^{2\pi i u}-1}\right)$$

The R.H.-factorization problem also boils down to finding fns $\eta^\pm$ s.t.

$$\eta^-(u) - \eta^+(u) = \frac{1}{(z-\delta)^{n+1}} \qquad \delta\in\mathbb{C}^\times,\ n\geq 0$$

Let $d\in\mathbb{C}$ be s.t. $\delta = e^{2\pi i d}$

Solution: $$\eta^\pm_{d,r} = -\frac{(-\delta)^{-r}}{2}\prod_{k=1}^{r-1}\left(\frac{\partial_u}{2\pi i k}+1\right)\cdot\left(\frac{\psi^\pm(u-d)}{\pi i}\pm\frac{1}{2}\right)$$

(19.8) Fundamental Solns:

$$\phi^+(u) = e^{-\gamma A_0} A(u)^{-1}\prod_{n\geq 1} A(u+n)^{-1} e^{A_0/n} \qquad \phi^-(u) = e^{-\gamma A_0}\prod_{n\geq 1} A(u-n)\, e^{A_0/n}$$

Monodromy matrix

$$S(u) = \lim_{N\to\infty}\left(A(u+N)\,A(u+N-1)\cdots A(u-N)\right)$$

work as long as $[A_0, A(u)] = 0$.