Lecture 2.

Finite and (untwisted) affine cases

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$\mathfrak{g}$ = finite dimensional simple Lie algebra / $\mathbb{C}$

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra $\quad \Delta = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ simple roots

$\overset{\vee}{\Delta} = \{h_i\}_{i\in I} \subset \mathfrak{h}$ simple coroots

$A = (a_{ij})_{i,j\in I}$ Cartan matrix $\quad D$ = Diagonal matrix $(d_i)_{i\in I}$ where

$d_i \in \mathbb{N}^\times$ and $\gcd(d_i : i\in I) = 1$.

$(\cdot,\cdot) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ fixed non-degenerate symmetric invariant bilinear form.

Its restriction to $\mathfrak{h}$ is again nondegenerate and gives rise to iso.

$\nu : \mathfrak{h} \to \mathfrak{h}^* \quad : \quad \nu(d_i h_i) = \alpha_i \quad (\forall\, i\in I)$. We use $(\cdot,\cdot)$ to denote

the form on $\mathfrak{h}^*$: $\quad (\alpha_i, \alpha_j) = d_i a_{ij} \quad \forall\, i,j \in I$.

$R = R_- \sqcup R_+ \subset \mathfrak{h}^*$ root system $\quad (R_+ = R \cap Q_+ ,\ R_- = -R_+$

where $\quad Q_+ = \sum_{i\in I} \mathbb{N}\alpha_i \subset \mathfrak{h}^*$).

$Q = \sum \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ (root lattice) $\qquad \overset{\vee}{Q} = \sum \mathbb{Z} h_i \subset \mathfrak{h}$ (coroot lattice)

Fundamental reflections $\quad s_i \in GL(\mathfrak{h}^*) \ : \ s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$

$$= \lambda - \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$$

$W \subset GL(\mathfrak{h}^*)$ group generated by $\{s_i\}_{i\in I}$

(Weyl group) $\qquad$ it preserves $(\cdot,\cdot)$

Recall DA is positive definite $\Rightarrow W$ is finite.

$\forall\, i,j\in I \quad m_{ij}$ = order of $s_i s_j$ is given by

$a_{ij}a_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

(2.1) Properties of the Weyl group.

Recall: we have length function $\ell(w) = \min\{r \mid w = s_{i_1} \cdots s_{i_r} ; i_1 \dots i_r \in I\}$

$\ell = \ell(w)$ $\quad s_{i_1} \cdots s_{i_\ell} = w$ is called a reduced expression of w.

Partial order on $\mathfrak{h}^*$: $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$

(1) W preserves R. $\quad R = \bigcup_{i \in I} W\alpha_i$ (hence $\dim \mathfrak{g}_\alpha = 1$, $\forall \alpha \in R$)

Proof. We have defined operators $\tilde{w} \in \mathrm{Aut}\,\mathfrak{g}$ s.t. $\tilde{w}\,\mathfrak{g}_\alpha = \mathfrak{g}_{w\alpha}$. Hence W preserves R. Now assume $\alpha = \sum_{i \in I} n_i \alpha_i \in R_+$ be of min'l height ($ht(\alpha) = \sum n_i$) s.t. $\alpha \notin \bigcup_{i \in I} W\alpha_i$. Then $s_i(\alpha)$ is also not in this set. Hence $(\alpha, \alpha_i) \leq 0$ $\forall i$ $\Rightarrow$ $(\alpha, \alpha) \leq 0$ contradicts positive-def.

(2) Let $\alpha \in R_+$. Then $s_i(\alpha) < 0 \iff \alpha = \alpha_i$.

Proof. $\quad s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i < 0 \iff \alpha = \alpha_i$.

(3) $\ell(ws_i) < \ell(w) \iff w(\alpha_i) < 0 \iff$ for any red exp. $w = s_{i_1} \cdots s_{i_\ell}$ (exchange property)

$\exists\, j \quad 1 \leq j \leq \ell$ s.t. $\quad s_{i_j} \cdots s_{i_\ell} = s_{i_{j+1}} \cdots s_{i_\ell} s_i$

Proof. $w(\alpha_i) < 0 \Rightarrow$ Exchange property : let $\beta_j = s_{i_{j+1}} \cdots s_{i_\ell} \alpha_i$ $\quad (j = 0, \dots, \ell)$

Then $\beta_0 < 0$ and $\beta_\ell > 0$ $\Rightarrow \exists\, j$ s.t. $\beta_j > 0$ and $\beta_{j-1} < 0$. But

$\beta_{j-1} = s_{i_j}(\beta_j)$ so $\beta_j = \alpha_{i_j}$, i.e. $\alpha_{i_j} = s_{i_{j+1}} \cdots s_{i_\ell}(\alpha_i)$ $\quad$ (if $\alpha = u\beta$, $s_\alpha = u s_\beta u^{-1}$, $\alpha, \beta \in \mathfrak{h}^*$, $u \in W$)

$\Rightarrow \quad s_{i_j} = s_{i_{j+1}} \cdots s_{i_\ell} s_i s_{i_\ell} \cdots s_{i_{j+1}}$ $\quad \square$

Exchange property $\Rightarrow \ell(ws_i) < \ell(w)$ clear.

$\ell(ws_i) < \ell(w) \Rightarrow w(\alpha_i) < 0$: Assume $w(\alpha_i) > 0$. Then $ws_i(\alpha_i) < 0$ and by previous implication $\ell(ws_is_i) < \ell(ws_i)$ $\quad \square$

(4) W admits the following presentation :

generators : s_i ($i \in I$) $\qquad$ relations : $s_i^2 = 1$ $\forall i \in I$

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} = s_j s_i s_j \cdots \quad \forall i, j \in I$$

<u>Proof</u>. We have the following consequence of Exchange Property

<u>Lemma</u> Let M be a monoid with 1 and $T_i \in M$ $(\forall i)$ s.t.

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots \quad (m_{ij} \text{ terms}).$$

For $w = s_{i_1} \cdots s_{i_\ell}$ red. exp., $T_w := T_{i_1} \cdots T_{i_\ell}$ is indep. of the choice of red. exp.

Using this lemma we can prove the presentation of W. Let G be an arbitrary group, $f : I \to G$ a set map s.t. $f(s_i)^2 = 1$ and $(f(s_i) f(s_j))^{m_{ij}} = 1$. By Lemma we have a set map $\tilde{f} : W \to G$. It is easy to see that $\tilde{f}$ is a group hom. Thus W satisfies the universal property of the group presented as above.

Proof of Lemma. Let $\underline{s} = s_{i_1} \cdots s_{i_\ell}$ and $\underline{s}' = s_{j_1} \cdots s_{j_\ell}$ be two red. exp. of w. We prove $T_{i_1} \cdots T_{i_\ell} = T_{j_1} \cdots T_{j_\ell}$ by induction on ℓ. $\ell = 1$ is clear. If $i_1 = j_1$ or $i_\ell = j_\ell$ we are done by induction. Otherwise $\ell(w s_{i_\ell}) < \ell(w)$ and $\ell(w s_{j_\ell}) < \ell(w)$. By exchange property we can replace $\underline{s}'$ by $s_{j_1} \cdots \hat{s}_{j_t} \cdots s_{j_\ell} s_{i_\ell}$ (similarly $\underline{s}$). If $t \neq 1$ we obtain an exp. of w whose 1^{st} term agree with $\underline{s}'$ and last term with $\underline{s}$, and we are done by induction. Otherwise

$$\underline{s} \rightsquigarrow \underline{s}_1 = s_{i_2} \cdots s_{i_\ell} s_{j_\ell} \quad \ldots \text{continuing} \rightsquigarrow \cdots s_{j_\ell} s_{i_\ell} s_{j_\ell}$$

$$\underline{s}' \rightsquigarrow \underline{s}'_1 = s_{j_2} \cdots s_{j_\ell} s_{i_\ell} \quad \rightsquigarrow \cdots s_{i_\ell} s_{j_\ell} s_{i_\ell}$$

↑ T's are same by assumption.

and we are done by hypothesis.

(5) Let $C = \{ h \in \mathfrak{h}_{\mathbb{R}} : \alpha_i(h) \geq 0 \ \forall i \}$

Then (a) C is fundamental domain for action of W on $\mathfrak{h}_{\mathbb{R}}$.

(b) $\mathfrak{h}_{\mathbb{R}} = \bigcup_{w \in W} w(C)$. Hence $\exists!\, w_0 \in W$ s.t. $w_0 C = -C$ (longest element)

Proof. (a) Let $h \in C$ and $w(h) = h' \in C$. $w = s_{i_1} \cdots s_{i_\ell}$.
Then $\alpha_{i_\ell}(h) \geq 0 \Rightarrow w\alpha_{i_\ell}(h') \geq 0$. But $w\alpha_{i_\ell} < 0$. Hence
$\alpha_{i_\ell}(h) = 0 \Rightarrow s_{i_\ell}(h) = h$.
This also proves that if $w \in W_h$ then $s_{i_1}, \ldots, s_{i_\ell} \in W_h$.
(b) For $h \in \mathfrak{h}_{\mathbb{R}}$ let $h' \in W \cdot h$ be (the largest) element [maximal]. Then $h' \in C$.

(6) $\exists!\ \theta \in R_+$ s.t. $\theta + \alpha_i \notin R_+$ (maximal root).

$$m = \frac{(\theta,\theta)}{2} = 3 \text{ for } G_2;\ 2 \text{ for } B, C, F;\ 1 \text{ for rest.}$$

Proof. Since $s_i(\theta)$ is a root we get $(\theta, \alpha_i) \geq 0\ \forall i$. Let $\theta = \sum_{i \in I} n_i \alpha_i$, $n_i \in \mathbb{N}$. Let $J = \{i \in I \mid n_i = 0\}$. Then $(\alpha_i, \alpha_j) = 0\ \forall j \in J, i \notin J$ contradicting indecomposibility of A. Now if β is another max'l root we get
$(\theta, \beta) = \sum n_i (\alpha_i, \beta) > 0 \Rightarrow \theta - \beta$ is a root (or zero). If it is positive or negative we contradict maximality of θ or β. Hence $\beta = \theta$. □

(2.2) Some more notations: $\{e_i, f_i, h_i\}_{i \in I}$ generators of $\mathfrak{g}$.

For $\alpha \in \mathfrak{h}^*$ define $\alpha^\vee = \frac{2\nu^{-1}(\alpha)}{(\alpha,\alpha)} \in \mathfrak{h}$.

$\rho \in \mathfrak{h}^*$ is chosen so that $\rho(h_i) = 1\ \forall i$.

Fundamental weights $\omega_i \in \mathfrak{h}^*$ s.t. $\omega_i(h_j) = \delta_{ij}\ \forall i, j \in I$.

Fundamental coweights $\omega_i^\vee \in \mathfrak{h}$ s.t. $\alpha_i(\omega_j^\vee) = \delta_{ij}\ \forall i, j \in I$

$P = \sum_{i \in I} \mathbb{Z}\omega_i \subset \mathfrak{h}^*$ weight lattice

$P^\vee = \sum \mathbb{Z}\omega_i^\vee \subset \mathfrak{h}$ coweight lattice

Note $Q \subset P$ and $Q^\vee \subset P^\vee$.

(2.3) Finite to affine. Let $L\mathfrak{g} := \mathfrak{g}[z, z^{-1}]$. Let $x(n) = x \cdot z^n$ for every $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Then $L\mathfrak{g}$ (is infinite-dim'l) Lie algebra with bracket $[x(k), y(l)] = [x,y](k+l)$. Define $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c$ its central extension:

$$[x(k), y(l)] = [x,y](k+l) + mk\,\delta_{k+l,0}\,\overset{(x,y)}{c} \quad , \quad ad(c) \equiv 0$$

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d \quad : \quad [d, x(n)] = n\,x(n) \qquad [d,c] = 0 .$$

$\tilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. We extend $(\cdot,\cdot)$ to $\tilde{\mathfrak{h}}$ by declaring $(c,d) = \frac{1}{m}$ and $(c,h) = 0 = (d,h) = (c,c) = (d,d)$. This gives an iso. $\tilde{\nu} : \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}^*$. Let $\delta, \Lambda_0 \in \tilde{\mathfrak{h}}^*$ be linear forms dual to d and c resp. Then $\tilde{\nu}$ is determined by

$$\tilde{\nu}|_{\mathfrak{h}} = \nu \qquad \tilde{\nu}(md) = \Lambda_0 \qquad \tilde{\nu}(mc) = \delta$$

We have root space decomposition of $\tilde{\mathfrak{g}}$ relative to $\tilde{\mathfrak{h}}$

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \Big(\bigoplus_{\hat{\alpha} \in \hat{R}} \tilde{\mathfrak{g}}_{\hat{\alpha}} \Big) \quad \text{where}$$

$$\hat{R} = \{\alpha + n\delta : \text{ either } \alpha \in R,\ n \in \mathbb{Z} \text{ or } \alpha = 0,\ n \in \mathbb{Z}^\times\}$$

Define
$$\alpha_0 = -\theta + \delta . \qquad h_0 = -\theta^\vee + c$$

Then $\hat{\Delta} = \{\alpha_i\}_{i \in I} \cup \{\alpha_0\}$, $\hat{\Delta}^\vee = \{h_i\}_{i \in I} \cup \{h_0\}$ gives realization of $\hat{A} = (a_{ij})_{i,j \in \hat{I}}$, $\hat{I} = I \cup \{0\}$ where $a_{i0} = -\theta(h_i)$, $a_{0i} = -\alpha_i(\theta^\vee)$

Set $e_0 = f_\theta(+1)$ and $f_0 = e_\theta(-1)$ where $e_\theta \in \mathfrak{g}_\theta$, $f_\theta \in \mathfrak{g}_{-\theta}$ are chosen so that $(e_\theta, f_\theta) = \frac{1}{m}$. Then $[e_0, f_0] = -\theta^\vee + c$.

$\tilde{\mathfrak{g}}$ is Kac-Moody algebra associated with $\hat{A}$ and symmetrizing matrix $D = (d_i)_{i \in \hat{I}}$, $d_0 = m$. Non-deg. form:

$(\cdot,\cdot)|_{\tilde{\mathfrak{h}}}$ as above. $\quad (x(k), y(l)) = (x,y)\,\delta_{k+l,0}$.

(2.4) Affine Weyl group. $W^a :=$ group generated by reflections $\{s_i : i \in \hat{I}\} \subset GL(\tilde{\mathfrak{h}})$ or $GL(\tilde{\mathfrak{h}}^*)$

Let us compute the action of s_0. Recall $\alpha_0 = -\theta + \delta$ and $h_0 = -\theta^\vee + c$.

$s_0(\tilde{h}) = \tilde{h} - \alpha_0(\tilde{h}) h_0 \qquad s_0(\tilde{\lambda}) = \tilde{\lambda} - \tilde{\lambda}(h_0)\alpha_0 \qquad \forall\ \tilde{h} \in \tilde{\mathfrak{h}},\ \tilde{\lambda} \in \tilde{\mathfrak{h}}^*$

Then we get for $\tilde{h} = h + kd + \ell c \in \mathfrak{h} \oplus \mathbb{C}d \oplus \mathbb{C}c = \tilde{\mathfrak{h}}$

$$s_0(h + kd + \ell c) = (h - (\theta(h) - k)\theta^\vee) + kd + (\ell - \theta(h) - k)c$$

Similarly $s_0(\lambda + k\Lambda + \ell\delta) = (\lambda - (\lambda(\theta^\vee) - k)\theta) + k\Lambda + (\ell - \lambda(\theta^\vee) - k)\delta$

Since W^a fixes (pointwise) $\mathbb{C}c \subset \tilde{\mathfrak{h}}$, we work with the quotient space $\tilde{\mathfrak{h}}/\mathbb{C}c$. Further we fix affine hyperplane $\delta = 1$ in this quotient space

$$\begin{array}{ccc} \mathfrak{h} & \simeq & \{\delta = 1\} \subset \tilde{\mathfrak{h}}/\mathbb{C}c \\ \psi & & \\ h & \longmapsto & h + d \pmod{\mathbb{C}c} \end{array} \qquad (*)$$

Under this identification we have the following: W^a acts on $\mathfrak{h}$ by affine linear transformations where W acts as usual and $s_0(h) = h - (\theta(h) - 1)\theta^\vee$ is reflection in the affine hyperplane $\{\theta = 1\}$

Prop. $W^a \simeq W \ltimes Q^\vee$.

Proof. Set $t_{\theta^\vee} = s_0 s_\theta : h \longmapsto s_0(h - \theta(h)\theta^\vee)$

$$= (h - \theta(h)\theta^\vee) - (\theta(h - \theta(h)\theta^\vee) - 1)\theta^\vee$$
$$= h + \theta^\vee.$$

Clearly $w t_{\theta^\vee} w^{-1} = t_{w\theta^\vee}$ $\forall\ w \in W$; and $t_{\alpha^\vee} t_{\beta^\vee} = t_{\alpha^\vee + \beta^\vee}$.

Thus $\begin{array}{ccc} Q^\vee & \longrightarrow & W^a \\ \alpha^\vee & \longmapsto & t_{\alpha^\vee} \end{array}$ realizes $Q^\vee$ as normal subgroup of W^a.

Now $W^a/Q^\vee \simeq W$ and W^a is generated by W and $Q^\vee$; $W \cap Q^\vee = \{1\}$ □

Under the identification (*) each real root $\alpha + n\delta$ defines an affine linear function $\alpha + n\delta\,(h) = \alpha(h) + n$. Let $H_{\alpha,n} \subset \mathfrak{h}_{\mathbb{R}}$ be corresponding affine hyperplane: $H_{\alpha,n} = \{h \in \mathfrak{h}_{\mathbb{R}} : \alpha(h) + n = 0\}$

$C := \{h \in \mathfrak{h}_{\mathbb{R}} : \alpha_i(h) > 0 \ \forall i, \ \theta(h) < 1\}$ is a connected component of $\mathfrak{h}_{\mathbb{R}} \setminus \bigcup H_{\alpha,n}$

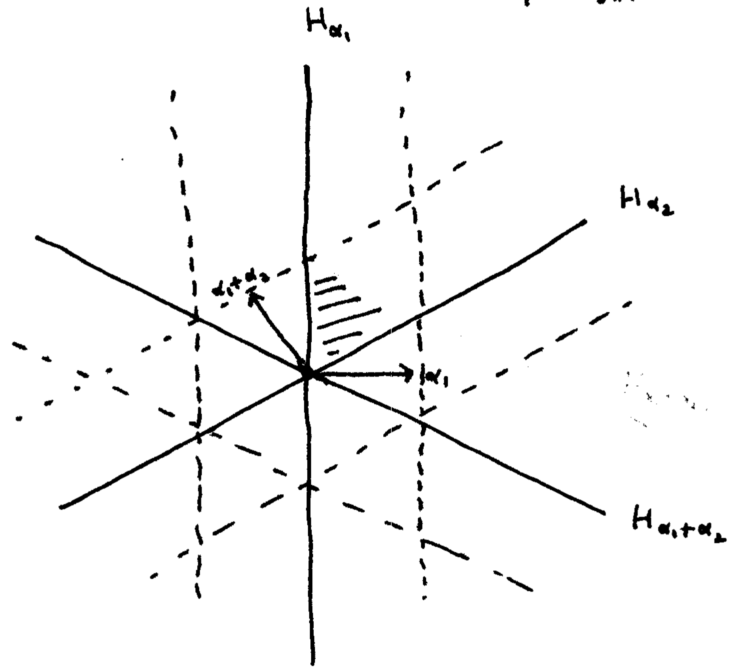
(Fundamental alcove)

Examples: A_2

$t_{h_1+h_2} = s_0 s_1 s_2 s_1$
$= s_0 s_2 s_1 s_2$

$t_{h_1} = s_2 s_0 s_2 s_1$

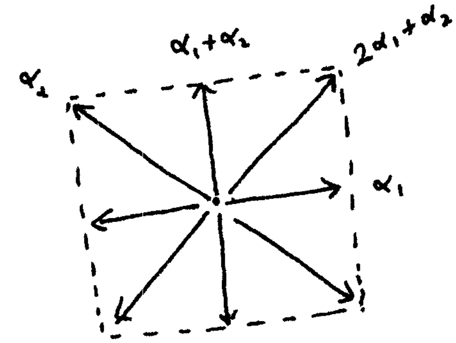
$t_{h_2} = s_1 s_0 s_1 s_2$



Picture of affine hyperplane arrangement in $\mathfrak{h}$ ($\alpha_i \leftrightarrow h_i$)

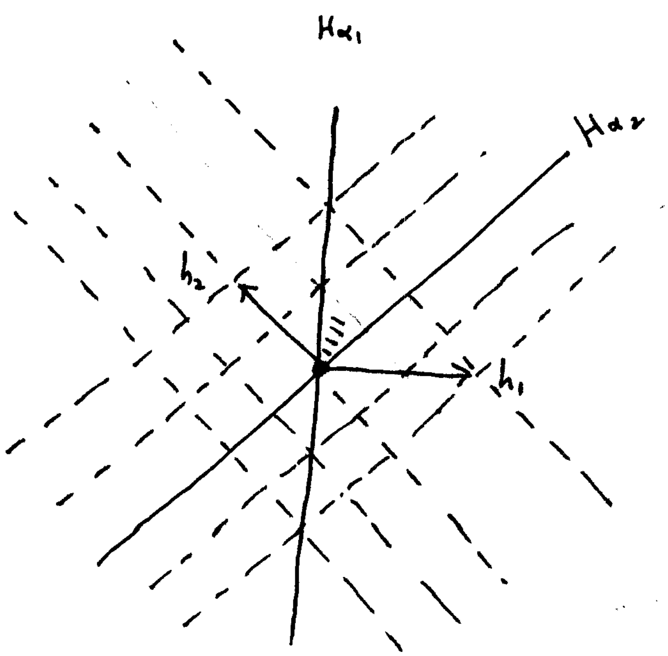
B_2 $\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$

Root system:



$\omega_1^\vee = h_1 + h_2$

$\omega_2^\vee = \frac{1}{2}h_1 + h_2$



$s_\theta = s_1 s_2 s_1$

$t_{h_1+h_2} = s_0 s_1 s_2 s_1$

$t_{h_2} = s_1 s_0 s_1 s_2$

$t_{h_1} = s_0 \underline{s_1 s_2 s_1 s_2 s_1} s_0 s_1$

$= s_0 s_2 s_1 s_2 s_1$

(2.5) Extended affine Weyl group : $W^e := W \ltimes P^\vee$

where $P^\vee$ is the coweight lattice $P^\vee = \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z}) = \bigoplus_{i \in I} \mathbb{Z}\, \omega_i^\vee$ $\quad (\alpha_i(\omega_j^\vee) = \delta_{ij})$

Then $W^a \subset W^e$ is a subgroup and $\Pi = W^e / W^a \simeq P^\vee / Q^\vee$ is a finite abelian group.

Length function : $\ell(w) = \# \{ \alpha \in \hat{R}_+ \mid w(\alpha) \in \hat{R}_- \}$

$= \#$ walls between C and $w(C)$

Π can be identified with elements of length 0. It permutes walls of $\overline{C}$ naturally labelled by $\hat{I}$. Thus Π permutes elements of $\hat{I}$ preserving $\hat{A}$.

Hence Π acts on W^a, $\pi s_i \pi^{-1} = s_{\pi(i)}$ $\quad \forall \pi \in \Pi \quad i \in \hat{I}$.

Prop. $W^e \simeq \Pi \ltimes W^a$. There is a bijection

$$J = \{ i \in I \mid \theta(\omega_i^\vee) = 1 \} \longleftrightarrow \Pi \setminus \{1\}$$

$$j \longmapsto \pi_j = t_{\omega_j^\vee} w_j w_0$$

w_j = longest element of root system obtained by deleting j^{th} vertex.

Proof (1) We begin by proving that π_j has length 0. Let $a \in C$ and $b = \pi_j(a)$. We have to show that $b \in C$:

• for $k \neq j$. $\alpha_k(b) = \alpha_k\left(w_j w_0(a) + \omega_j^\vee\right) = \alpha_k\left(w_j w_0(a)\right)$

$= (w_j \alpha_k)(w_0 a) > 0$

• $\alpha_j(b) = 1 + w_j(\alpha_j)(w_0 a) \geq 1 + \theta(w_0 a) > 0$

since $\theta - w_j(\alpha_j) \geq 0$, $w_0 a \in -C$ and $\theta(w_0 a) < -1$.

Finally write $\theta = \sum_{i \in I} n_i \alpha_i$, $n_i > 0$ and $n_j = 1$.

Now $w_j \theta = \alpha_j + \cdots$ is again positive. Hence $\theta(b) = 1 + w_j \theta(w_0 a) < 1$.

(2) Let $\gamma \in \Pi \setminus \{1\}$. Write $\gamma = t \cdot w$ where $t \in P^\vee$, $w \in W$.
Then $t \neq 1$ since $P^\vee \cap W = \{1\}$. Now $\gamma(0) = t(0) \in \overline{C} \cap P^\vee$.
Claim: $P^\vee \cap \overline{C} = \{\omega_j^\vee : j \in J\}$ is clear; since $\omega^\vee = \sum k_i \omega_i^\vee \in \overline{C}$ implies $k_i \in \mathbb{N}$ and $\sum_{i \in I} k_i n_i \leq 1$ where $\theta = \sum_{i \in I} n_i \alpha_i$. So $\gamma(0) = \omega_i^\vee$ for some $i \in J$

Hence $\pi_i^{-1} \gamma(0) = 0 \Rightarrow \pi_i^{-1} \gamma = 1$

(2.5) Braid group (extended affine)

Define B^e to be group generated by $T(w)$ $w \in W^e$ subject to relations
$$T(w_1 w_2) = T(w_1) T(w_2) \quad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$

First presentation: $W^e = \Pi \ltimes W^a$ (Π acts on $\hat{I} = I \cup \{0\}$ as symmetries of affine Dynkin diagram)

Generators: $U_\pi = T(\pi)$ $\forall \pi \in \Pi$

$T_i = T(s_i)$ $\forall i \in \hat{I}$

Relations: $T_i T_j T_i \cdots = T_j T_i T_j \cdots$ m_{ij} terms $i, j \in \hat{I}$

$U_\pi U_{\pi'} = U_{\pi'} U_\pi$ $\quad U_\pi T_i U_\pi^{-1} = T_{\pi(i)}$

Second presentation $W^e = W \ltimes P^\vee$

Generators $Y_{\lambda^\vee}$ ($\lambda^\vee \in P^\vee$) defined as $\begin{cases} Y_{\lambda^\vee} := T(t_{\lambda^\vee}) & \text{for } \lambda^\vee \text{ dominant} \\ Y_{\mu^\vee - \nu^\vee} := T(t_{\mu^\vee}) T(t_{\nu^\vee})^{-1} & (\mu^\vee, \nu^\vee \text{ dominant}) \end{cases}$

$T_i = T(s_i)$ $i \in I$

Relations $\{T_i\}_{i \in I}$ satisfy braid relations

$\{Y_{\lambda^\vee}\}_{\lambda^\vee \in P^\vee}$ commute

$T_i Y_{\lambda^\vee} T_i^{-1} = Y_{\lambda^\vee}$ if $\alpha_i(\lambda^\vee) = 0$

$$T_i^{-1} Y_{\lambda^\vee} T_i^{-1} = \begin{cases} Y_{s_i \lambda^\vee} \\ \text{if } \alpha_i(\lambda^\vee) = 1 \end{cases}$$