Lecture 20

(20.0) Definition / Construction of the functor $\Gamma : \operatorname{Rep}^{NC} Y \longrightarrow \operatorname{Rep} \mathcal{U}$

$$Y = Y_\hbar \mathfrak{g} \qquad \text{Yangian of } \mathfrak{g}$$

$$\mathcal{U} = \mathcal{U}_q(L\mathfrak{g}) \qquad \text{quantum loop algebra of } \mathfrak{g}$$

Assume $\hbar \in \mathbb{C} \setminus \mathbb{Q}$ and $q = e^{\pi i \hbar}$. Hence q is not a root of unity.

$\operatorname{Rep}^{NC} Y$ is defined to be the full subcategory of $\operatorname{Rep} Y$ consisting of V s.t.

$$a, b \in \sigma(V) \Rightarrow a - b \notin \mathbb{Z}^\times.$$

Consider $V \in \operatorname{Rep}^{NC} Y$. The action of Y on V is governed by $\operatorname{End} V$-valued rational functions $\{\xi_j(u), x_j^\pm(u)\}_{j \in I}$. Consider the difference equation (abelian)

$$\phi_j(u+1) = \xi_j(u)\,\phi_j(u)$$

By the general theory, we have
- 2 fundamental solutions $\phi_j^\pm(u)$
- monodromy matrix $S_j(u)$

(1) Let $\psi_j(z) := S_j(u)\big|_{z = e^{2\pi i u}}$: $\operatorname{End}(V)$-valued rational fn. of z, regular at 0 and ∞ s.t. $\psi_j(0)^{-1} = \psi_j(\infty) = e^{\pi i \hbar \xi_{j,0}} = q^{\xi_{j,0}}$

(2) Set $g_j^+(u) = \phi_j^+(u+1)^{-1}$ and $g_j^-(u) = \phi_j^-(u)$. Explicitly

$$g_j^+(u) = e^{\gamma \hbar \xi_{j,0}} \prod_{n \geq 1} \xi_j(u+n)\, e^{-\hbar \xi_{j,0}/n}$$

$$g_j^-(u) = e^{-\gamma \hbar \xi_{j,0}} \prod_{n \geq 1} \xi_j(u-n)\, e^{\hbar \xi_{j,0}/n}$$

Choose scalars $c_j^\pm \in \mathbb{C}^\times$ s.t. $c_j^+ c_j^- = d_j \Gamma(\hbar d_j)^2$ and define ($\forall k \in \mathbb{Z}$)

$$E_{j,k} = c_j^+ \oint_C e^{2\pi i k u} g_j^+(u)\, x_j^+(u)\, du \qquad F_{j,k} = c_j^- \oint_C e^{2\pi i k u} g_j^-(u)\, x_j^-(u)\, du$$

where C is a contour enclosing $\sigma(V)$ and none of its $\mathbb{Z}^{\times}$-translates.
Such a contour can be chosen since V is non-congruent.

In particular $g_j^{\pm}(u)$ are holomorphic within C.

Thm. (1) and (2) define ~~the~~ an action of $\mathcal{U}$ on V. Namely, we have the following commutation relations (see Lecture 8)

(let $\psi^{\pm}_{j,\pm r}$ be the coefficients of the Taylor series expansion of $\psi_j(z)$ near $z=\infty$ and 0 :

$$\psi_j(z) = \sum_{r\geq 0} \psi^+_{j,r} z^{-r} \quad \text{near } \infty$$
$$= \sum_{r \geq 0} \psi^-_{j,-r} z^{r} \quad \text{near } 0 \Bigg)$$

(QL1) $$[\psi_{j_1}(z_1), \psi_{j_2}(z_2)] = 0 \quad \forall j_1, j_2 \in I,$$

(QL23) $$\mathrm{Ad}(\psi_{j_1}(z))\left(z E_{j_2,\ell} - q_{j_1}^{a_{j_1j_2}} E_{j_2,\ell+1}\right) = z\, q_{j_1}^{a_{j_1j_2}} E_{j_2,\ell} - E_{j_2,\ell+1}$$

$\forall j_1, j_2 \in I$, $\ell \in \mathbb{Z}$. (similarly for F : replace $a_{j_1j_2}$ by $-a_{j_1j_2}$)

(QL4) $$E_{j_1,\ell_1+1} E_{j_2,\ell_2} - q_{j_1}^{a_{j_1j_2}} E_{j_2,\ell_2} E_{j_1,\ell_1+1} = q_{j_1}^{a_{j_1j_2}} E_{j_1,\ell_1} E_{j_2,\ell_2+1} - E_{j_2,\ell_2+1} E_{j_1,\ell_1}$$

(QL5) $$[E_{j_1,\ell_1}, F_{j_2,\ell_2}] = \delta_{j_1j_2} \frac{\psi^+_{j_1,\ell_1+\ell_2} - \psi^-_{j_1,\ell_1+\ell_2}}{q_{j_1} - q_{j_1}^{-1}}$$

and q-Serre relations.

Remark: we don't have to check q-Serre relations since they follow from others ~~by~~ on f.d. repns (or more generally integrable repns.).
(see §20.2 below)

(20.1) Example : $\mathfrak{g} = \mathfrak{sl}_2$ $\qquad V = V(\lambda, b) \qquad (\lambda \in \mathbb{N}, \ b \in \mathbb{C})$

A basis of $V(\lambda, b)$: $m_\lambda(r) \quad (0 \leq r \leq \lambda)$

$$x^+(u)\, m_\lambda(r) = \frac{\hbar(\lambda - r + 1)}{u - b + (r-1)\hbar}\, m_\lambda(r-1)$$

$$\xi(u)\, m_\lambda(r) = \frac{(u-b-\hbar)(u-b+\lambda\hbar)}{(u-b+(r-1)\hbar)(u-b+r\hbar)}\, m_\lambda(r)$$

$$x^-(u)\, m_\lambda(r) = \frac{\hbar(r+1)}{u-b+r\hbar}\, m_\lambda(r+1)$$

$$\phi^+(u)\, m_\lambda(r) = \frac{\Gamma(u-b-\hbar)\,\Gamma(u-b+\lambda\hbar)}{\Gamma(u-b+(r-1)\hbar)\,\Gamma(u-b+r\hbar)}\, m_\lambda(r)$$

$$\phi^-(u) = \frac{\Gamma^-(u-b+(r-1)\hbar)\,\Gamma^-(u-b+r\hbar)}{\Gamma^-(u-b-\hbar)\,\Gamma^-(u-b+\lambda\hbar)} \quad (\text{on } m_\lambda(r))$$

$$\| $$

$$g^-(u) \qquad (g^+ = \text{same expression with } \Gamma^+)$$

$$\Gamma^\pm(x) := \Gamma(1 \pm x) \qquad \text{Take} \quad c^+ = c^- = \Gamma(\hbar)$$

$$\psi(z)\, m_\lambda(r) = \frac{(z - q^2\beta)(z - q^{-2\lambda}\beta)}{(z - q^{-2(r-1)}\beta)(z - q^{-2r}\beta)}\, q^{\lambda - 2r}\, m_\lambda(r) \qquad \beta = e^{2\pi i b}$$

$$E_k\, m_\lambda(r+1) = \Gamma(\hbar) \oint_C e^{2\pi i k u} \frac{\Gamma^+(u-b+(r-1)\hbar)\,\Gamma^+(u-b+r\hbar)}{\Gamma^+(u-b-\hbar)\,\Gamma^+(u-b+\lambda\hbar)}\, \frac{\hbar(\lambda - r)}{u-b+r\hbar}\, du \ \ m_\lambda(r) \qquad (r \geq 0)$$

$$= \Gamma(\hbar)\, q^{-2r}\, \beta\, \frac{\Gamma^+(-\hbar)\,\Gamma^+(0)\,\hbar(\lambda-r)}{\Gamma^+(-r\hbar-1)\,\Gamma^+((\lambda-r)\hbar)}\, m_\lambda(r)$$

(20.2) Reduction of q-Serre relations

$i \neq j \in I \qquad m = 1 - a_{ij}$.

$$S_{ij}(k_1 \ldots k_m; l) = \sum_{\pi \in S_m} \sum_{r=0}^{m} (-1)^r \begin{bmatrix} m \\ r \end{bmatrix} E_{i,k_{\pi(1)}} \cdots E_{i,k_{\pi(r)}}\, E_{j,l}\, E_{i,k_{\pi(r+1)}} \cdots E_{i,k_{\pi(m)}}$$

(1) $S_{ij}(0, \ldots, 0; l) \ (\forall l \in \mathbb{Z}) \implies S_{ij}(k_1 \ldots k_m; l) \ \forall k_1, \ldots, k_m, l \in \mathbb{Z}$

(use $\displaystyle [H_{i,r}, E_{j,k}] = \frac{[r a_{ij}]_{q_i}}{r} E_{j,k+r} \quad \forall r \in \mathbb{Z}^\times, \ k \in \mathbb{Z}$)

(ii) Let V be a f.d. repn. Consider $U_{q_i}sl_2$ - action on $\mathrm{End}\, V$
(e, f, k: generators)

$$f \cdot X = [F_{i,0}, X] \qquad e \cdot X = E_{i,0} X - \mathrm{Ad}(\psi_{i,0}^{+})(X)$$

$$k \cdot X = \mathrm{Ad}\, \psi_{i,0}(X)$$

Then (for any $l \in \mathbb{Z}$) $E_{j,l} \in \mathrm{End}\, V$ is a lowest weight vector of lowest weight a_{ij}. By $U_{q_i}sl_2$-representation theory $e^m \cdot E_{j,l} = 0$

$$\mathrm{LHS} = S_{ij}(0, \dots, 0; l) = 0.$$

(20.3) For the proof of Theorem (20.0), let us restrict to $\mathfrak{g} = sl_2$ (solely for notational convenience).

<u>Proposition.</u> Let C be a contour with interior domain D. $\Omega_1, \Omega_2 \overset{\text{open}}{\subset} \mathbb{C}$ containing $\overline{D}$; $f : \Omega_1 \times \Omega_2 \to \mathrm{End}\, V$ hol. fn. s.t. $[\xi(u), f(u,v)] = 0$. Then

(1) If $u \notin \overline{D} \pm \hbar$ then

$$(\mathrm{Ad}\, \xi(u))^{\pm 1} \oint_C f(u,v)\, x^+(v)\, dv = \oint \left(\frac{u-v+\hbar}{u-v-\hbar}\right)^{\pm 1} f(u,v)\, x^+(v)\, dv$$

(for x^- replace $\hbar$ by $-\hbar$)

(2) If $u \notin \overline{D} \pm \hbar - \mathbb{N}^\times$

$$\mathrm{Ad}\,(g^+(u))^{\pm 1} \oint_C f(u,v)\, x^+(v)\, dv = \oint_C \left(\frac{\Gamma^+(u-v-\hbar)}{\Gamma^+(u-v+\hbar)}\right)^{\pm 1} f(u,v)\, x^+(v)\, dv$$

(for x^-: replace $\hbar$ by $-\hbar$; for g^-: replace Γ^+ by Γ^- (& $-\mathbb{N}^\times$ by $+\mathbb{N}^\times$))

(3) If $u \notin \overline{D} \pm \hbar + \mathbb{Z}$

$$\mathrm{Ad}\, S_i(u)^{\pm} \oint_C f(u,v)\, x^+(v)\, dv = \oint_C \left(\frac{q^2 z - w}{z - q^2 w}\right)^{\pm 1} f(u,v)\, x^+(v)\, dv$$

$$q = e^{\pi i \hbar} \qquad z = e^{2\pi i u} \qquad w = e^{2\pi i v}.$$

Proof. We have: $\xi(u)\, x^+(v)\, \xi(u)^{-1} = \dfrac{u-v+\hbar}{u-v-\hbar}\, x^+(v) - \dfrac{2\hbar}{u-v-\hbar}\, x^+(u-\hbar)$

Multiply by $f(u,v)$ on the left and integrate over C (on variable v) $\square$

(20.4) QL23 :

$$\mathrm{Ad}\,\psi(z)\left(z E_\ell - q^2 E_{\ell+1}\right) \overset{?}{=} q^2 z E_\ell - E_{\ell+1} .$$

$\|$ by Prop. (20.3) $\qquad\qquad \|$ by defn.

$$z w^\ell \left(\frac{q^2 z - w}{\frac{w}{z} - q^2 w}\right) - q^2 w^{\ell+1}\left(\frac{q^2 z - w}{z - q^2 w}\right) \overset{?}{=} q^2 z w^\ell - w^{\ell+1}$$

$$\equiv \quad z\left(q^2 z - w\right) - q^2 w \left(q^2 z - w\right) = \left(q^2 z - w\right)\left(z - q^2 w\right) \quad \checkmark$$

(20.5) QL4 :

$$E_{k+1} E_\ell - q^2 E_k E_{\ell+1} = q^2 E_\ell E_{k+1} - E_{\ell+1} E_k$$

Prop (20.3) $\Rightarrow$ $\qquad (\forall\, v \notin \bar{D} - \hbar - \mathbb{N}^\times)$

$$\oint e^{2\pi i (ku+\ell v)} \left(e^{2\pi i u} - e^{2\pi i (v+\hbar)}\right) g^+(u)\, x^+(u)\, du \cdot g^+(v)$$

$$= g^+(v) \oint e^{2\pi i (ku+\ell v)} \left(e^{2\pi i u} - e^{2\pi i (v+\hbar)}\right) \frac{\Gamma^+(v-u+\hbar)}{\Gamma^+(v-u-\hbar)}\, g^+(u)\, x^+(u)\, du$$

zeroes and poles cancel !

Hence this equation is true for all v.

$$\text{L.H.S.} = (c^+)^2 \oint\oint e^{2\pi i (ku+\ell v)} \left(e^{2\pi i u} - e^{2\pi i (v+\hbar)}\right) \frac{\Gamma^+(v-u+\hbar)}{\Gamma^+(v-u-\hbar)}\, g^+(u)\, g^+(v)\, x^+(u)\, x^+(v)\, du\, dv$$

$$\text{R.H.S.} = (c^+)^2 \oint\oint e^{2\pi i (ku+\ell v)} \left(e^{2\pi i (u+\hbar)} - e^{2\pi i v}\right) \frac{\Gamma^+(u-v+\hbar)}{\Gamma^+(u-v-\hbar)}\, g^+(u)\, g^+(v)\, x^+(v)\, x^+(u)\, du\, dv$$

Use $\quad x^+(v)\, x^+(u) = \dfrac{v-u+\hbar}{v-u-\hbar}\, x^+(u)\, x^+(v) \mp \dfrac{\hbar}{v-u-\hbar}\left(x^+(u)^2 + x^+(v)^2\right).$

and $\quad \Gamma^+(x)\,\Gamma^-(x) = \dfrac{2\pi i x}{e^{\pi i x} - e^{-\pi i x}}$.

(20.6) QLS : let us just check it for $k = l = 0$

$$[E_0, F_0] = \frac{\psi_0^+ - \psi_0^-}{q - q^{-1}}$$

$$\text{L.H.S.} = \underset{c^+c^-}{\boxed{\Gamma(\hbar)^2}} \left[\oint\!\!\oint g^+(u)\,x^+(u)\,g^-(v)\,x^-(v) - \oint\!\!\oint g^-(v)\,x^-(v)\,g^+(u)\,x^+(u) \right]$$

→ Problem : we cannot use Prop. (20.3) directly, since we can have 2 points within C which differ by $\hbar + \mathbb{N}^\times$. [Validity of Prop: $u \notin \overline{D} + \hbar - \mathbb{N}^\times$, $v \notin \overline{D} - \hbar + \mathbb{N}^\times$]

To avoid this, define

$$I(\delta) := \oint\!\!\oint g^+(u+\delta)\,x^+(u)\,g^-(v-\delta)\,x^-(v) - \oint\!\!\oint g^-(v-\delta)\,x^-(v)\,g^+(u+\delta)\,x^+(u)$$

↳ hol. in a disc $|\delta| < R$ where R is s.t. $g^\pm$ are hol. within $D \pm \delta'$ $\forall\,\delta' ;\ |\delta'| < R$.

By making C small enough, we can find $r < R$ s.t.

$(D - \delta) \cap (D - \hbar + \mathbb{N}^\times) = \phi$.

$\forall\, \delta$ with $|\delta| > r$

Within $r < |\delta| < R$ we can carry out the following computation

$$I(\delta) = \oint\!\!\oint g^+(u+\delta)\,g^-(v-\delta) \left(\frac{\Gamma^-(v-\delta-u+\hbar)}{\Gamma^-(v-\delta-u-\hbar)}\, x^+(u)\,x^-(v) - \frac{\Gamma^+(u+\delta-v-\hbar)}{\Gamma^+(u+\delta-v+\hbar)}\, x^-(v)\,x^+(u) \right) du\,dv$$

equal since $\Gamma^+(-x) = \Gamma^-(x)$

$$= \oint\!\!\oint g^+(u+\delta)\,g^-(v-\delta)\, \frac{\Gamma^+(u+\delta-v-\hbar)}{\Gamma^+(u+\delta-v+\hbar)}\, \frac{\hbar}{u-v}\,\big(\xi(v) - \xi(u)\big)\,du\,dv$$

$z = e^{2\pi i u}$

$dz = 2\pi i\, e^{2\pi i u}\,du$

$$= \oint g^+(u+\delta)\,g^-(u-\delta)\, \frac{\Gamma^+(\delta-\hbar)}{\Gamma^+(\delta+\hbar)}\, \hbar\, \xi(u)\,du$$

↳ this is hol. within $|\delta| < R$ as well

$$\text{L.H.S.} = \Gamma(\hbar)^2\, \hbar\, \frac{\Gamma^-(\hbar)}{\hbar\,\Gamma(\hbar)} \oint \frac{g^+(u)\,g^-(u)\,\xi(u)\,du}{S(u)} = \frac{2\pi i}{q - q^{-1}} \oint \frac{z^{-1}\,\psi(z)\,dz}{2\pi i} = \psi_0^+ - \psi_0^-$$