Lecture 3
Quasi-triangular Hopf algebras

(3.0) Our next goal is to associate to each Kac-Moody algebra a quasi-triangular Hopf algebra. For this topic we return back to general Kac-Moody case which will be specialized to finite / affine cases laters.

(3.1) Quasi-triangular Hopf algebra. A Hopf algebra H is a unital associative $\mathbb{C}$-algebra together with linear maps

$$\underset{\text{coproduct}}{\Delta : H \to H \otimes H} \qquad \underset{\text{antipode}}{S : H \to H} \qquad \underset{\text{counit}}{\varepsilon : H \to \mathbb{C}}$$

satisfying the following axioms: • $\Delta \otimes 1 \circ \Delta = 1 \otimes \Delta \circ \Delta$ [co-associativity]

- Δ is an algebra homomorphism $\quad \varepsilon$ is an algebra hom.
- $(\varepsilon \otimes \mathrm{Id})\Delta(x) = 1 \otimes x$ and $(1 \otimes \varepsilon)\Delta(x) = x \otimes 1 \quad \forall x \in H$ [counit axiom]
- $S(xy) = S(y)S(x) \quad \forall x, y \in H$
- $\mu(S \otimes 1)\Delta(x) = \varepsilon(x) \cdot 1 = \mu(1 \otimes S)\Delta(x) \quad \forall x \in H$

where $\mu : H \otimes H \to H$ is the multiplication

Examples: (1) Let $\mathfrak{g}$ be a Lie algebra, $\mathcal{U}(\mathfrak{g})$ its enveloping algebra

Recall $\quad \mathcal{U}(\mathfrak{g}) = T\mathfrak{g} \Big/ \langle x \otimes y - y \otimes x = [x,y] \rangle \qquad T\mathfrak{g} = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$ (tensor algebra)

$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \forall x \in \mathfrak{g}$. $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra

$S(x) = -x$

$\varepsilon(x) = 0$

(2) G : a finite group $\quad \mathbb{C}G$ = group algebra of G

$\Delta(g) = g \otimes g \qquad S(g) = g^{-1} \qquad \varepsilon(g) = 1 \quad \forall g \in G.$

Note: in the examples above $\Delta(x) = \Delta^{21}(x)$. In this case we say Δ (or H) is cocommutative.

(3.2) Let $(H, \Delta, S, \varepsilon)$ be a Hopf algebra. A quasi-triangular structure on H is an element (invertible) $R \in H \otimes H$ s.t.

- $\Delta^{21}(x) = R\, \Delta(x)\, R^{-1} \quad \forall\, x \in H$
- $(\Delta \otimes 1)(R) = R_{13} R_{23}$
  $(1 \otimes \Delta)(R) = R_{13} R_{12}$ (Cabling identities)

Recall: for $X = a \otimes b$
$X_{13} = a \otimes 1 \otimes b$
and so on

Prop. (1) $(\varepsilon \otimes 1)(R) = 1 \otimes 1 = (1 \otimes \varepsilon)(R)$

(2) $(S \otimes 1)(R) = R^{-1} = (1 \otimes S^{-1})(R)$

Proof. (1) $\Delta \otimes 1\,(R) = R_{13} R_{23}$. Apply $\varepsilon \otimes 1 \otimes 1$ to get

$$(\varepsilon \otimes 1 \otimes 1)(\Delta \otimes 1)(R) = (\varepsilon \otimes 1 \otimes 1)(R_{13}) \cdot R_{23}$$

LHS $= 1 \otimes R = R_{23}$ by counit axiom. Thus $(\varepsilon \otimes 1 \otimes 1) R_{13} = 1 \otimes 1 \otimes 1$ gives $1 \otimes \varepsilon(R) = 1 \otimes 1$.

Similarly applying $1 \otimes \varepsilon \otimes 1$ to the same equation

(2). Apply $\mu_{12} \circ (S \otimes 1 \otimes 1)$ to $\Delta \otimes 1\,(R) = R_{13} R_{23}$ to get

$$\varepsilon \otimes 1\,(R) = (S \otimes 1)(R) \cdot R \Rightarrow (S \otimes 1)(R) = R^{-1} \text{ using (1).}$$

Now we apply $\mu_{23} \circ (S \otimes S \otimes 1)$ to $1 \otimes \Delta(R) = R_{13} R_{12}$ to get

$$(S \otimes 1)(R) \cdot (S \otimes S)(R) = 1 \otimes 1 \Rightarrow S \otimes S\,(R) = R.$$

Hence $R = (1 \otimes S)(S \otimes 1)(R) = 1 \otimes S\,(R^{-1})$ proving the last assertion. $\square$

Lemma. Cabling identities $\Rightarrow R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

Proof

$$R_{12}R_{13}R_{23}R_{12}^{-1} = R_{12}(\Delta\otimes 1)(R)R_{12}^{-1} = (\Delta^{op}\otimes 1)(R)$$

$$= R_{23}R_{13}\,. \quad \text{where } \Delta^{op}(x) = b\otimes a \text{ if } \Delta(x) = a\otimes b.$$

(or $\sigma\circ\Delta = \Delta^{op}$ where σ is flip). $\square$

(3.3) Braided tensor categories

Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. A tensor product on $\mathcal{C}$ is a $\mathbb{C}$-bilinear functor $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$. We say $\mathcal{C}$ is a <u>monoidal category</u> if it is equipped with a tensor product $\otimes$, a unit object $\mathbb{1}\in\mathcal{C}$ and natural transformations (isomorphisms)

- $a : (\cdot\otimes\cdot)\otimes\cdot \xrightarrow{\sim} \cdot\otimes(\cdot\otimes\cdot)$

(i.e. $\forall\, X,Y,Z\in\mathcal{C}$ we have iso $a_{X,Y,Z} : (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)$)

- $l : \mathbb{1}\otimes\cdot \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}}$ $\qquad r : \cdot\otimes\mathbb{1} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}}$

(left and right unit : $l_X : \mathbb{1}\otimes X \xrightarrow{\sim} X$ $\quad r_X : X\otimes\mathbb{1} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}}(X) = X$)

subject to following axioms

[Unit axiom]

$$\begin{tikzcd}
(X\otimes\mathbb{1})\otimes Y \arrow[rr, "a_{1,X,Y}"] \arrow[dr, "r_X\otimes \mathrm{id}_Y"'] & & X\otimes(\mathbb{1}\otimes Y) \arrow[dl, "\mathrm{id}_X\otimes l_Y"] \\
& X\otimes Y &
\end{tikzcd}$$

commutes

[Pentagon axiom]

$\forall\, X,Y,Z,W\in\mathcal{C}$

$$\begin{tikzcd}
& (X\otimes(Y\otimes Z))\otimes W \arrow[r] & X\otimes((Y\otimes Z)\otimes W) \arrow[dd] \\
((X\otimes Y)\otimes Z)\otimes W \arrow[ur] \arrow[dr] & & \\
& (X\otimes Y)\otimes(Z\otimes W) \arrow[r] & X\otimes(Y\otimes(Z\otimes W))
\end{tikzcd}$$

A monoidal category is <u>rigid</u> if we have an (anti-)autoequivalence of $\mathcal{C}$: $X \mapsto X^*$ and natural morphisms $ev_X : X^* \otimes X \to 1$, $\pi_X : 1 \to X \otimes X^*$ s.t. following diagrams commute.

$$\begin{tikzcd}
1 \otimes X \arrow[r] \arrow[d, "l_X"] & (X \otimes X^*) \otimes X \arrow[d] \\
X & \\
X \otimes 1 \arrow[u, "r_X"] & X \otimes (X^* \otimes X) \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
X^* \otimes 1 \arrow[r] \arrow[d] & X^* \otimes (X \otimes X^*) \\
X^* & \\
1 \otimes X^* \arrow[u] & (X^* \otimes X) \otimes X^* \arrow[l] \arrow[uu]
\end{tikzcd}$$

Example. Let $(H, \Delta, S, \varepsilon)$ be a Hopf algebra. Then $\mathrm{Rep}\, H$ is a rigid monoidal category. Moreover the associativity constraint is trivial due to the coassociativity axiom $\Delta \otimes 1 \circ \Delta = 1 \otimes \Delta \circ \Delta$.

Such categories are called <u>strict</u>. We will only work with strict monoidal categories and hence omit the associativity constraint.

Let $\mathcal{C}$ be a (strict) rigid monoidal category. A <u>commutativity constraint</u> or <u>braided structure</u> on $\mathcal{C}$ is a natural transformation (iso.)

$$c : \cdot \otimes \cdot \longrightarrow (\cdot \otimes \cdot) \circ \sigma \quad (\sigma \text{ is the flip})$$

i.e. $\forall\, X, Y \in \mathcal{C}$ we have $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ natural in X, Y.

satisfying

- $c_{X,1} = id_X = c_{X,1}$
- $\forall\, X, Y, Z \in \mathcal{C}$ we have the following commutative diagrams:

$$\begin{tikzcd}
 & X \otimes Y \otimes Z \arrow[dl, "id_X \otimes c_{Y,Z}"'] \arrow[dr, "c_{X \otimes Y, Z}"] & \\
X \otimes Z \otimes Y \arrow[rr, "c_{X,Z} \otimes id_Y"'] & & Z \otimes X \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \otimes Y \otimes Z \arrow[dl, "c_{X,Y} \otimes id_Z"'] \arrow[dr, "c_{X, Y \otimes Z}"] & \\
Y \otimes X \otimes Z \arrow[rr, "id_Y \otimes c_{X,Z}"'] & & Y \otimes Z \otimes X
\end{tikzcd}$$

Example. Let (H, R) be a quasi-triangular Hopf algebra

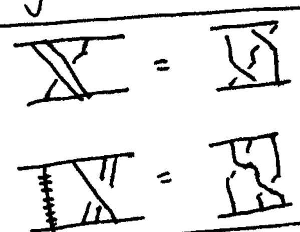
Define $\check{R}_{X,Y} : X \otimes Y \to Y \otimes X$ by $(X, Y \in \operatorname{Rep} H)$

$$\check{R}_{X,Y} = \sigma \circ R_{X,Y}.$$ Then $\check{R}_{X,Y}$ defines a braided structure on $\operatorname{Rep} H$.

- $\Delta^{21}(x) = R\,\Delta(x)\,R^{-1} \iff \check{R}_{X,Y}$ is a morphism of H-modules
- Cabling identities $\iff$ Commutativity of diagrams

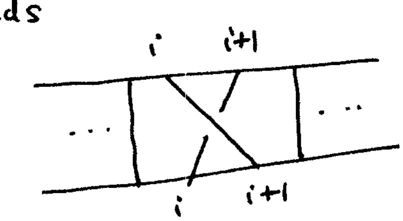
$$\begin{tikzcd} & X \otimes Y \otimes Z \arrow[dl] \arrow[dr] & \\ X \otimes Z \otimes Y \arrow[rr] & & Z \otimes X \otimes Y \end{tikzcd} \qquad \begin{tikzcd} & X \otimes Y \otimes Z \arrow[dl] \arrow[dr] & \\ Y \otimes X \otimes Z \arrow[rr] & & Y \otimes Z \otimes X \end{tikzcd}$$

Pictorially:



- Yang-Baxter equation $\implies$ For any $X \in \operatorname{Rep}(H)$, we have $B_n \circlearrowright X^{\otimes n}$ by $T_i \mapsto \check{R}_{i,i+1}$

where B_n = braid group on n-strands

= generators T_i =  $(1 \le i \le n-1)$

relations $T_i T_j = T_j T_i \qquad |i-j| \ge 2$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad (1 \le i \le n-2)$