Lecture 4 Quantum groups

(4.0) Recall : a quasi-triangular Hopf algebra is a pair (H, R) where H is a Hopf algebra. $R \in (H\otimes H)^{\times}$ satisfying

$$\Delta^{21}(x) = R\,\Delta(x)\,R^{-1} \qquad \Delta\otimes 1(R) = R_{13}R_{23}$$
$$1\otimes\Delta(R) = R_{13}R_{12}$$

Last time we proved that $S\otimes 1(R) = R^{-1} = (1\otimes S^{-1})(R)$ and $\varepsilon\otimes 1(R) = 1\otimes 1 = 1\otimes\varepsilon(R)$

Thm Let $u = \mu\circ(S\otimes 1)(R_{21})$. Then [u is called Drinfeld element]

(a) $S^2(x) = u\,x\,u^{-1} \quad \forall\, x\in H$

(b) $u^{-1} = \mu\circ(S^{-1}\otimes S)(R_{21})$

Proof. (a) Let $x\in H$ and let us write $\Delta^{(3)}(x) = (\Delta\otimes 1)(\Delta(x)) = (1\otimes\Delta)(\Delta(x)) = \sum_k f_k\otimes g_k\otimes h_k$.

For convenience let us write $R = a_i\otimes b_i$

$R\otimes 1\cdot\Delta^{(3)}(x) = ((12)\circ\Delta^{(3)}(x))\,R\otimes 1$ can be written as

$$a_i f_k\otimes b_i g_k\otimes h_k = g_k a_i\otimes f_k b_i\otimes h_k$$

Apply $(13)\circ(1\otimes S\otimes S^2)$ to this equation and multiply the factors to get

$$S^2(h_k)\otimes S(g_k)S(b_i)\otimes a_i f_k = S^2(h_k)\otimes S(b_i)\,S(f_k)\otimes g_k a_i$$

$$S^2(h_k)\,S(g_k)\,S(b_i)\,a_i f_k = S^2(h_k)\,S(b_i)\,S(f_k)\,g_k\,a_i$$

$\Rightarrow$ Using $S(f_k)\,g_k\otimes h_k = 1\otimes x$ the RHS $= S^2(x)\cdot u$

Using $f_k\otimes g_k S(h_k) = x\otimes 1$ the LHS $= u\cdot x$ and we are done

(b) $v := S^{-1}(b_i)\,S(a_i)$ Then $uv = u\,S^{-1}(b_i)\,S(a_i) = S(b_i)\,u\,S(a_i)$

$$= S(b_j b_i)\,a_j\,S(a_i) = \mu\circ\sigma\circ(1\otimes S)\left(a_j\otimes b_j\cdot S(a_i)\otimes b_i\right) = 1$$

Since $S\otimes 1(R) = R^{-1}$. Finally $S^2(v)\,u = u\,v\,u^{-1}u = uv = 1$

$\Rightarrow$ u admits both left and right inverses $(= v)$ □

(4.1) Again let $A = (a_{ij})_{i,j\in I}$ be a generalized (symmetrizable) Cartan matrix

D = diagonal$(d_i : i \in I)$ symmetrizing matrix.

Definition : $\tilde{\mathcal{U}}$ = unital assoc. algebra generated by $h \in \mathfrak{h}$, e_i, f_i $(i \in I)$ subject to the following relations

$$[h, h'] = 0 \qquad [h, e_i] = \alpha_i(h)\, e_i \qquad [h, f_i] = -\alpha_i(h)\, f_i$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \qquad q_i = q^{d_i} \quad k_i = q_i^{h_i}$$

q = deformation parameter is either considered formal variable or a complex number not a root of unity. In either case $\ln(q) = t$ needs to be fixed.

Hopf structure
$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i$$
$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i$$
$$\Delta(h) = h \otimes 1 + 1 \otimes h$$

Easy exercise: Δ extends to algebra hom $\tilde{\mathcal{U}} \to \tilde{\mathcal{U}} \otimes \tilde{\mathcal{U}}$.

$$S(e_i) = -e_i k_i^{-1} \qquad S(f_i) = -k_i f_i \qquad S(h) = -h$$

Again S extends to algebra anti-homomorphism $\tilde{\mathcal{U}} \to \tilde{\mathcal{U}}$.

$$\varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(h) = 0$$

Remarks : (1) $e^a\, b\, e^{-a} = e^{\mathrm{ad}(a)} . b$. Thus we have

$$k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j \qquad k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j$$

In general for $\mu \in Q$, $\mu = \sum_{i \in I} n_i \alpha_i$ define $k_\mu = \prod_{i\in I} k_i^{n_i}$.

(2) Subalgebras of $\tilde{\mathcal{U}}$: $\tilde{\mathcal{U}}^+$ is gen. by e_i $(i \in I)$ $\tilde{\mathcal{U}}^-$ is gen by f_i $(i \in I)$.

$\tilde{\mathcal{U}}^0$ is gen. by $h \in \mathfrak{h}$ (actually $\tilde{\mathcal{U}}^0 = \mathrm{Sym}(\mathfrak{h})$)

$\tilde{\mathcal{U}}^{\geq} = \tilde{\mathcal{U}}^+ \tilde{\mathcal{U}}^0$ $\qquad \tilde{\mathcal{U}}^{\leq} = \tilde{\mathcal{U}}^- \tilde{\mathcal{U}}^0$ (also $= \tilde{\mathcal{U}}^0 \tilde{\mathcal{U}}^-$)

(also $= \tilde{\mathcal{U}}^0 \tilde{\mathcal{U}}^+$)

$\tilde{\mathcal{U}}^{\geq}$ and $\tilde{\mathcal{U}}^{\leq}$ are Hopf subalgebras. $\tilde{\mathcal{U}}^+ \otimes \tilde{\mathcal{U}}^0 \otimes \tilde{\mathcal{U}}^- \xrightarrow{\sim} \tilde{\mathcal{U}}$ vector space iso.

(3) $\tilde{\mathcal{U}}^{\pm} = \bigoplus_{\mu \in Q_+} \tilde{\mathcal{U}}^{\pm}_{\pm\mu} \qquad \tilde{\mathcal{U}}^{\pm}_{\pm\mu} = \{ x \in \tilde{\mathcal{U}}^{\pm} \mid [h, x] = \pm\mu(h)\, x \}$

and $\dim \tilde{\mathcal{U}}^{\pm}_{\pm\mu} < \infty$.

(4) $\forall\, x \in \tilde{\mathcal{U}} \qquad \Delta(S(x)) = S \otimes S\ \Delta^{op}(x)$ and similarly for S^{-1}.

(5) Notation $\Delta^{(n)} : \tilde{\mathcal{U}} \to \tilde{\mathcal{U}}^{\otimes n} \quad (n \geq 1)$

$\Delta^{(1)} = id \qquad \Delta^{(2)} = \Delta \qquad \Delta^{(n)} = \Delta \otimes 1 \circ \Delta^{(n-1)}$ (unambiguously defined by coassoc.)

(4.2) Drinfeld pairing: $\langle \cdot , \cdot \rangle : \tilde{\mathcal{U}}^{\geq} \times \tilde{\mathcal{U}}^{\leq} \to \mathbb{C}$ (or $t^{-1}\mathbb{C}[[t]]$)

$\langle h, h' \rangle = \frac{1}{t}(h, h') \qquad \langle e_i, f_j \rangle = \dfrac{\delta_{ij}}{q_i - q_i^{-1}} \qquad \langle e_i, h \rangle = \langle h, f_i \rangle = 0$

$\langle 1, \cdot \rangle = \langle \cdot , 1 \rangle = \varepsilon(\cdot)$

$\langle aa', b \rangle = \langle a \otimes a', \Delta^{21}(b) \rangle \qquad \langle a, bb' \rangle = \langle \Delta(a), b \otimes b' \rangle$

$\langle \cdot , \cdot \rangle$ preserves weights, in particular it defines pairing between

$$\tilde{\mathcal{U}}^{+}_{\mu} \times \tilde{\mathcal{U}}^{-}_{-\mu} \to \mathbb{C} \quad (\text{or } t^{-1}\mathbb{C}[[t]]).$$

(4.3) Example of $\mathfrak{sl}_2$. generators h, e, f

(1) $\langle e^n, f^n \rangle = q^{-\frac{n(n-1)}{2}} \dfrac{[n]!}{(q - q^{-1})^n}$ where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

$[n]! = [n][n-1] \dots [1]$.

We prove this by induction on n. $n=1$ being the definition of $\langle e, f \rangle$.

$$\langle e^n, f^n \rangle = \langle \Delta(e^n), f \otimes f^{n-1} \rangle$$

$\Delta(e^n) = (e \otimes k + 1 \otimes e)^n$. For weight reasons only term pairing non-trivially is the one containing single e in the first $\otimes$ factor

i.e. $\quad e \otimes \sum_{i=0}^{n-1} e^i k e^{n-1-i} = \left(\sum_{i=0}^{n-1} q^{-2i} \right) e \otimes k \cdot e^{n-1}$

$$= \frac{1-q^{-2n}}{1-q^{-2}}\, e\otimes k e^{n-1} = q^{-n+1}[n]\, e\otimes k e^{n-1}$$

Thus $\langle e^n, f^n\rangle = q^{-n+1}[n]\,\langle e, f\rangle\langle k e^{n-1}, f^{n-1}\rangle$.

Next we claim that $\langle k e^{n-1}, f^{n-1}\rangle = \langle e^{n-1}, f^{n-1}\rangle$. This is true since

$$\langle k e^{n-1}, f^{n-1}\rangle = \langle k\otimes e^{n-1}, \Delta^{21}(f^{n-1})\rangle \quad \text{and } \Delta(f^{n-1}) = (f\otimes 1 + k^{-1}\otimes f)^{n-1}$$

$$= \langle k, 1\rangle\langle e^{n-1}, f^{n-1}\rangle = \langle e^{n-1}, f^{n-1}\rangle \qquad = f^{n-1}\otimes 1 + \cdots$$

$$(\langle k, 1\rangle = \varepsilon(k) = \varepsilon(q^h) = 1)$$

Hence $\langle e^n, f^n\rangle = +q^{-n+1}\dfrac{[n]}{q-q^{-1}}\langle e^{n-1}, f^{n-1}\rangle$ and the result follows.

(2) $\langle h^a, h^b\rangle = \delta_{ab}\, a!\,\left(\frac{2}{t}\right)^a$

Let us assume $a \leq b$ (the other case is treated similarly). Then

$\langle h^a, h^b\rangle = \langle \Delta^{(b)}(h^a), h\otimes h\otimes\cdots\otimes h\rangle$. Now

$$\Delta^{(b)}(h^a) = \left[\sum_{i=0}^{b-1} 1^{\otimes i}\otimes h\otimes 1^{\otimes b-1-i}\right]^a.$$ We only get a non-trivial pairing if $a=b$ and corresponding term is $a!\,(h\otimes\cdots\otimes h)$

Hence $\langle h^a, h^b\rangle = \delta_{ab}\, a!\,\langle h, h\rangle^a = \delta_{ab}\, a!\left(\frac{2}{t}\right)^a$.

(3) Finally $\langle h^a e^b, h^c f^d\rangle = \delta_{ac}\delta_{bd}\langle h^a, h^a\rangle\langle e^b, f^b\rangle$.

In particular $\langle\cdot,\cdot\rangle$ is non-degenerate. Its canonical tensor element is

$$R = \sum_{a,b\geq 0} (a!)^{-1}\left(\frac{t}{2}\right)^a \frac{q^{\frac{b(b-1)}{2}}}{[b]!}(q-q^{-1})^b\, h^a e^b\otimes h^a f^b$$

$$= e^{\frac{t}{2}(h\otimes h)}\cdot\sum_{b\geq 0}\frac{q^{\frac{b(b-1)}{2}}}{[b]!}\left((q-q^{-1})\, e\otimes f\right)^b = q^{\frac{h\otimes h}{2}}\exp_q\left((q-q^{-1})\, e\otimes f\right)$$

(4.4) In general, define $\mathrm{rad}^{\geq} \subseteq \tilde{\mathcal{U}}^{\geq}$ (and $\mathrm{rad}^{\leq} \subseteq \tilde{\mathcal{U}}^{\leq}$) to be subspace of elements $x \in \tilde{\mathcal{U}}^{\geq}$ s.t. $\langle x, y\rangle = 0$ $\forall\, y \in \tilde{\mathcal{U}}^{\leq}$ (resp. $\langle x, y\rangle = 0$ $\forall\, x \in \tilde{\mathcal{U}}^{\geq}$)

Similarly $\mathrm{rad}^{\pm}_{\pm\mu} \subset \tilde{\mathcal{U}}^{\pm}_{\pm\mu}$ $\forall\, \mu \in Q_+$ and $\mathrm{rad}^{\pm} = \bigoplus_{\mu \in Q_+} \mathrm{rad}^{\pm}_{\pm\mu}$.

Then $\mathrm{rad}^{\geq} = \mathrm{rad}^{+} \cdot \tilde{\mathcal{U}}^{0}$ and $\mathrm{rad}^{\leq} = \mathrm{rad}^{-} \cdot \tilde{\mathcal{U}}^{0}$

Lemma. $\mathrm{rad}^{\geq/\leq} \subset \tilde{\mathcal{U}}^{\geq/\leq}$ are Hopf ideals, i.e. two sided ideals s.t.

$$\Delta(\mathrm{rad}^{\geq/\leq}) \subset \mathrm{rad}^{\geq/\leq} \otimes \tilde{\mathcal{U}}^{\geq/\leq} + \tilde{\mathcal{U}}^{\geq/\leq} \otimes \mathrm{rad}^{\geq/\leq}$$

Proof. Let us prove this for $\geq$ only. For $x \in \mathrm{rad}^{\geq}$ and $a \in \mathcal{U}^{\geq}$, $b \in \mathcal{U}^{\leq}$ we have

$$\left.\begin{aligned} \langle ax, b\rangle &= \langle a \otimes x, \Delta^{21}(b)\rangle = 0 \\ \langle xa, b\rangle &= \langle x \otimes a, \Delta^{21}(b)\rangle = 0 \end{aligned}\right\} \Rightarrow ax, xa \in \mathrm{rad}^{\geq}$$

Similarly $\langle \Delta(x), b_1 \otimes b_2\rangle = \langle x, b_1 b_2\rangle = 0$ $\forall\, b_1, b_2 \in \tilde{\mathcal{U}}^{\leq}$

$$\Rightarrow \Delta(x) \in \mathrm{rad}^{\geq} \otimes \tilde{\mathcal{U}}^{\geq} + \tilde{\mathcal{U}}^{\geq} \otimes \mathrm{rad}^{\geq}$$ □

(4.5) We will prove that $\mathrm{rad}^{\geq} \cdot \tilde{\mathcal{U}}^{-}$ (and $\tilde{\mathcal{U}}^{+} \cdot \mathrm{rad}^{\leq}$) are two-sided ideals

For this we need to introduce derivations r_i, r_i' ($\forall\, i \in I$) of $\tilde{\mathcal{U}}^{\geq}$

Defn. For $x \in \mathcal{U}^{+}_{\mu}$ ($\mu \in Q_+$) define $r_i(x)$, $r_i'(x) \in \mathcal{U}^{+}_{\mu-\alpha_i}$ as:

$$\Delta(x) = x \otimes k_\mu + \sum_{i\in I} r_i'(x) \otimes e_i k_{\mu-\alpha_i} + \dots + \sum_{i \in I} e_i \otimes r_i(x) k_i + 1 \otimes x$$

Prop. (a) $r_i(e_j) = r_i'(e_j) = \delta_{ij}$ $\forall\, i,j \in I$

(b) $\forall\, x \in \mathcal{U}^{+}_{\mu}$, $x' \in \mathcal{U}^{+}_{\mu'}$

$$r_i(xx') = q^{(\mu, \alpha')} r_i(x) x' + x\, r_i(x')$$

$$r_i'(xx') = r_i'(x) x' + q^{(\mu, \alpha_i)} x\, r_i'(x')$$

(c) $$[x, f_i] = \frac{r_i(x) k_i - k_i^{-1} r_i'(x)}{q_i - q_i^{-1}} \quad \forall\, x \in \tilde{\mathcal{U}}^{+}$$

Proof. (a) $\Delta(e_j) = e_j \otimes k_j + 1 \otimes e_j \Rightarrow r_i(e_j) = r_i'(e_j) = \delta_{ij}$

(b) $\Delta(xx') = \Delta(x)\,\Delta(x')$ implies

$$\Delta(xx') = xx' \otimes k_{\mu+\mu'} + \sum_{i\in I} r_i'(xx') \otimes e_i k_{\mu+\mu'-\alpha_i} + \dots + \sum_{i \in I} e_i \otimes r_i(xx') k_i + 1 \otimes xx'$$

$$= \Delta(x)\cdot\Delta(x') = \Big(x \otimes k_\mu + \sum r_i'(x) \otimes e_i k_{\mu-\alpha_i} + \dots + \sum e_i \otimes r_i(x) k_i + 1 \otimes x\Big)\cdot$$
$$\Big(x' \otimes k_{\mu'} + \sum r_i'(x') \otimes e_i k_{\mu'-\alpha_i} + \dots + \sum e_i \otimes r_i(x') k_i + 1 \otimes x'\Big)$$

$$= xx' \otimes k_{\mu+\mu'} + \sum_i \Big(r_i'(x)x' \otimes e_i k_{\mu+\mu'-\alpha_i} + q^{(\mu,\alpha_i)} x r_i'(x') \otimes e_i k_{\mu+\mu'-\alpha_i}\Big) + \dots$$
$$+ \sum_i e_i \otimes \Big(q^{(\mu',\alpha_i)} r_i(x) x' + x r_i(x')\Big) k_i + 1 \otimes xx'$$

(c) For $x = e_j$ the claim follows from (a). For $x \in \mathcal{U}^+_\mu$, $x' \in \mathcal{U}^+_{\mu'}$

$$[xx', f_i] = [x, f_i]\,x' + x\,[x', f_i]$$
$$= \frac{1}{q_i - q_i^{-1}} \Big[r_i(x) k_i x' - k_i^{-1} r_i'(x) x' + x r_i(x') k_i - x k_i^{-1} r_i'(x') \Big]$$
$$= \frac{1}{q_i - q_i^{-1}} \Big(\big(q^{(\mu',\alpha_i)} r_i(x) x' + x r_i(x')\big) k_i - k_i^{-1}\big(r_i'(x) x' + q^{(\mu,\alpha_i)} x r_i'(x')\big)\Big)$$
$$= \frac{1}{q_i - q_i^{-1}} \big(r_i(xx') k_i - k_i^{-1} r_i'(xx') \big) \quad \text{if the claim holds for } x, x'.$$

□

(4.6) Prop. $\mathrm{rad}^+ \cdot \tilde{\mathcal{U}}^0 \cdot \tilde{\mathcal{U}}^-$ is a two-sided ideal of $\tilde{\mathcal{U}}$.
(similarly $\tilde{\mathcal{U}}^+ \cdot \tilde{\mathcal{U}}^0 \cdot \mathrm{rad}^-$)

Proof Let $x \in \mathrm{rad}^+_\mu$. Then $\forall\, y \in \tilde{\mathcal{U}}^-_{-\mu+\alpha_i}$ we have

$$0 = \langle x, f_i y\rangle = \langle e_i, f_i \rangle \langle r_i(x) k_i, y\rangle$$
$$0 = \langle x, y f_i \rangle = \langle r_i'(x), y\rangle \langle e_i k_{\mu-\alpha_i}, f_i\rangle$$

$\Rightarrow$ $r_i(x)$ and $r_i'(x) \in \mathrm{rad}^+_{\mu-\alpha_i}$ (or just 0).

Hence $[x, f_i] \in \mathrm{rad}^+_{\mu-\alpha_i}$ and we are done (see Prop 4.5 (c)) $\square$

(4.7) Let $\mathrm{rad} \subset \tilde{\mathcal{U}}$ be the two sided ideal generated by $\mathrm{rad}^\pm$.

Defn. $\mathcal{U}_q\mathfrak{g} := \tilde{\mathcal{U}}/\mathrm{rad}$.

By Prop 4.6 we have $\mathcal{U}_q\mathfrak{g} = \underbrace{\tilde{\mathcal{U}}^+/\mathrm{rad}^+}_{\mathcal{U}^+} \otimes \underbrace{\tilde{\mathcal{U}}^0}_{\mathcal{U}^0} \otimes \underbrace{\tilde{\mathcal{U}}^-/\mathrm{rad}^-}_{\mathcal{U}^-}$

Moreover $\langle \cdot , \cdot \rangle$ descends to a non-degenerate pairing between $\mathcal{U}^\geq$ and $\mathcal{U}^\leq$. Define

$$R \in \mathcal{U}^\geq \otimes \mathcal{U}^\leq \subset \mathcal{U}_q\mathfrak{g} \hat{\otimes} \mathcal{U}_q\mathfrak{g}$$

canonical element, i.e. if $\{A_\ell\}$ and $\{B_\ell\}$ are homogeneous basis of $\mathcal{U}^\geq$ and $\mathcal{U}^\leq$, dual to each other, then $R = \sum_\ell A_\ell \otimes B_\ell$.

Thm. $(\mathcal{U}_q\mathfrak{g}, R)$ is a quasi-triangular Hopf algebra. Let $u \in \mathcal{U}_q\mathfrak{g}$ be its Drinfeld element $u = \sum S(B_\ell) A_\ell$, and let $\rho \in \mathfrak{h}^*$ be s.t. $(\rho, \alpha_i) = \frac{(\alpha_i, \alpha_i)}{2} = d_i \quad \forall i \in I$. Then $C^q = q^{2\rho} u^{-1} \in \mathcal{U}_q\mathfrak{g}$ is central (called q-Casimir element)

Proof. We begin by proving cabling identities

$$\Delta \otimes 1 \, (R) = R_{13} R_{23} \quad (\in \mathcal{U}^\geq \otimes \mathcal{U}^\geq \otimes \mathcal{U}^\leq)$$

Pairing both sides with $B_k \otimes B_\ell \otimes A_s$ gives

$$\text{L.H.S.} = \langle \Delta(A_s), B_k \otimes B_\ell \rangle$$

$$\text{R.H.S.} = \langle A_s, B_k B_\ell \rangle$$

which are equal by Hopf property $\langle a, bb' \rangle = \langle \Delta(a), b \otimes b' \rangle$

Similarly the other cabling identity.

Next we need to prove that $\Delta^{21}(x) = R\,\Delta(x)\,R^{-1}$. This is clear for $x \in \mathcal{U}^0$ since R is zero weight, and $\Delta^{21} = \Delta$ on $\mathcal{U}^0$. Let us prove it for $x \in \mathcal{U}^+$. We need the following

Lemma. $\forall\, y \in \mathcal{U}^{\leq}$, $x \in \mathcal{U}^{\geq}$ we have

$$yx = \langle \bar{S}^{-1}(x^{(1)}), y^{(1)} \rangle \langle x^{(3)}, y^{(3)} \rangle\, x^{(2)} y^{(2)}, \quad \text{where } x^{(1)} \otimes x^{(2)} \otimes x^{(3)} = \Delta^{(3)}(x)$$

We continue with the proof of the theorem.

$$\begin{aligned} R\,\Delta(x) &= A_i\, x^{(1)} \otimes B_i\, x^{(2)} \\ &= A_i\, x^{(1)} \otimes x^{(3)} B_i^{(2)} \langle \bar{S}^{-1}(x^{(2)}), B_i^{(1)} \rangle \langle x^{(4)}, B_i^{(3)} \rangle \end{aligned}$$

$$\left[A_i \otimes B_i^{(1)} \otimes B_i^{(2)} \otimes B_i^{(3)} = 1 \otimes \Delta^{(3)} R = R_{14} R_{13} R_{12} = A_i A_j A_k \otimes B_k \otimes B_j \otimes B_i \right]$$

$$\begin{aligned} &= A_i A_j A_k\, x^{(1)} \otimes x^{(3)} B_j \langle \bar{S}^{-1}(x^{(2)}), B_k \rangle \langle x^{(4)}, B_i \rangle \\ &= x^{(4)} A_j\, \bar{S}^{-1}(x^{(2)})\, x^{(1)} \otimes x^{(3)} B_j = x^{(2)} \otimes x^{(1)} \cdot R \end{aligned}$$

Since $\bar{S}^{-1}(x_2)\, x_1 \otimes x_3 \otimes x_4 = 1 \otimes \Delta(x) = \cancel{} = 1 \otimes x^{(1)} \otimes x^{(2)}$. □

(4.8)
Proof of Lemma. $yx = \langle \bar{S}^{-1}(x^{(1)}), y^{(1)} \rangle \langle x^{(3)}, y^{(3)} \rangle\, x^{(2)} y^{(2)}$

One can easily check this on generators $y = h$ or f_j ; $x = h$ or e_i, for instance

$y = f_j$, $x = e_i$:

$$\Delta^{(3)}(e_i) = e_i \otimes k_i \otimes k_i + 1 \otimes e_i \otimes k_i + 1 \otimes 1 \otimes e_i$$

$$\bar{S}^{-1} \otimes 1 \otimes 1 \circ \Delta^{(3)}(e_i) = -k_i^{-1} e_i \otimes k_i \otimes k_i + 1 \otimes e_i \otimes k_i + 1 \otimes 1 \otimes e_i$$

$$\Delta^{(3)}(f_j) = f_j \otimes 1 \otimes 1 + k_j^{-1} \otimes f_j \otimes 1 + k_j^{-1} \otimes k_j^{-1} \otimes f_j$$

$$\text{RHS} = -\langle k_i^{-1} e_i, f_j \rangle k_i + e_i f_j + \langle e_i, f_j \rangle k_j^{-1} = e_i f_j \quad \text{if } j \neq i$$

$$= \frac{-k_i + k_i^{-1}}{q_i - q_i^{-1}} + e_i f_i \quad (\text{if } j = i) \qquad \left(\begin{aligned} \langle k_i^{-1} e_i, f_i \rangle &= \langle k_i^{-1} \otimes e_i, 1 \otimes f_i + f_i \otimes k_i^{-1} \rangle \\ &= \langle e_i, f_i \rangle = \frac{1}{q_i - q_i^{-1}} \end{aligned} \right)$$

Next we claim that this identity holds for $yz \in U^{\leq}$ if it holds for both y and z.

$$yzx = \langle \bar{S}^{-1}(x^{(1)}), z^{(1)} \rangle \langle x^{(3)}, z^{(3)} \rangle \; y x^{(2)} z^{(2)}$$

To apply it again for $y x^{(2)}$ we need to take $\Delta^{(3)}(x^{(2)})$

$$(1 \otimes \Delta^{(3)} \otimes 1)\, \Delta^{(3)}(x) = \Delta^{(5)}(x) \quad \text{and we get}$$

$$= \langle \bar{S}^{-1}(x^{(1)}), z^{(1)} \rangle \langle x^{(5)}, z^{(3)} \rangle \langle \bar{S}^{-1}(x_2), y^{(1)} \rangle \langle x^{(4)}, y^{(3)} \rangle \, x^{(3)} y^{(2)} z^{(2)}$$

$$= \langle \Delta(\bar{S}^{-1}(x^{(1)})), y^{(1)} \otimes z^{(1)} \rangle \langle \Delta(x^{(3)}), y^{(3)} \otimes z^{(3)} \rangle \; x^{(2)} (yz)^{(2)}$$

$$= \langle \bar{S}^{-1}(x^{(1)}), (yz)^{(1)} \rangle \langle x^{(3)}, (yz)^{(3)} \rangle \; x^{(2)} (yz)^{(2)}$$

Similarly one can show that this identity for x and $w \in \tilde{U}^{\geq}$ implies it for $w \cdot x \in U^{\geq}$. The Lemma is proved. $\square$