Lecture 5 Representations of $U_q\mathfrak{g}$

(5.0) Recall: last time we defined $U_q\mathfrak{g} = \tilde{U}/\text{radical}$. $R \in U_q\mathfrak{g}^{\otimes 2}$ the canonical element of the non-degenerate pairing $\langle\cdot,\cdot\rangle : U^{\geq} \times U^{\leq} \to \mathbb{C}$.

Thm. $(U_q\mathfrak{g}, R)$ is a quasi-triangular Hopf algebra

Let $u = \mu \circ (S \otimes 1)(R_{21})$ be the Drinfeld element. Then $S^2(x) = u x u^{-1}$ $\forall\, x \in U_q\mathfrak{g}$.

$$S^2(h) = h \qquad \begin{aligned} S^2(e_i) &= S(-e_i k_i^{-1}) = k_i e_i k_i^{-1} = q_i^2 e_i \\ S^2(f_i) &= S(-k_i f_i) = k_i f_i k_i^{-1} = q_i^{-2} f_i \end{aligned}$$

Choose $\rho \in \mathfrak{h}^*$ s.t. $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) = d_i$. Then above calculation shows that $S^2(x) = k_{2\rho}\, x\, k_{2\rho}^{-1} = q^{2\rho} x q^{-2\rho}$.

Define $C_q := q^{2\rho} u^{-1}$ (quantum Casimir element).

Cor. $C_q = q^{2\rho} u^{-1}$ is central.

(5.1) $R \in U_q\mathfrak{g}^{\otimes 2}$ has the form

$$R = q^{\sum x_i \otimes x^i} \cdot \sum a_\ell \otimes b_\ell$$

where $\{x_i\}$ is a basis of $\mathfrak{h}$, $\{x^i\}$ dual basis of $\mathfrak{h}$ w.r.t. $(\cdot,\cdot)$. $\{a_\ell\}$ is a homogeneous basis of $U^+ = \bigoplus U^+_\mu$ and $\{b_\ell\}$ (dual) basis of $U^- = \bigoplus U^-_{-\mu}$.

Proof: Let $n = |I|$ and $\{x_i\}_{i=1,\dots,n}$ be o.n. basis of $\mathfrak{h}$ wrt $(\cdot,\cdot)$.

Claim 1. $\langle x_1^{m_1} \cdots x_n^{m_n}, x_1^{m_1'} \cdots x_n^{m_n'} \rangle = \delta_{m_1 m_1'} \cdots \delta_{m_n m_n'} \dfrac{m_1! \cdots m_n!}{t^{m_1 + \cdots + m_n}}$

Proof is easy induction on $\min(\sum m_i, \sum m_i')$

Claim 2. For $p, p' \in \mathcal{U}^0$, $a \in \mathcal{U}^+_\mu$ $b \in \mathcal{U}^-_{-\mu}$ we have

$$\langle pa, p'b \rangle = \langle p, p' \rangle \langle a, b \rangle$$

Proof of claim 2: let us assume p and p' are monomials in $\{x_i\}$.

We first prove that $\langle pa, b \rangle = 0$ if $p \neq 1$. This is clear since

$$\langle pa, b \rangle = \langle p \otimes a, \Delta^{21}(b) \rangle = \langle p \otimes a, 1 \otimes b \rangle = \varepsilon(p) \langle a, b \rangle.$$

Finally $\langle pa, p'b \rangle = \langle \Delta(p) \Delta(a), p' \otimes b \rangle = \langle p \otimes a, p' \otimes b \rangle$

(since by weight reasons only relevant term of $\Delta(a)$ is $1 \otimes a$ and

$\Delta(p) = p \otimes 1 + \cdots$ where $\cdots$ has non-trivial monomials p'' on the second

factor and $\langle p'' a, b \rangle = 0$).

Hence if $\{a_\ell\}$ $\{b_\ell\}$ are basis (hgs) of $\mathcal{U}^+$ and $\mathcal{U}^-$, dual to each other

then
$$R = \sum_{\substack{m_1 \ldots m_n \geq 0 \\ \ell}} \frac{t^{m_1 + \cdots + m_n}}{m_1! \ldots m_n!} x_1^{m_1} \cdots x_n^{m_n} a_\ell \otimes x_1^{m_1} \cdots x_n^{m_n} b_\ell$$

$$= \exp\left(t \sum_i x_i \otimes x_i \right) \sum_\ell a_\ell \otimes b_\ell = q^{\sum_i x_i \otimes x_i} \sum_\ell a_\ell \otimes b_\ell \qquad \square$$

(5.2) For each $i \neq j \in I$, let $r = 1 - a_{ij}$ and define

$$\theta^+_{ij} = \sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix}_{q_i} e_i^{r-s} e_j e_i^s$$

$$\theta^-_{ij} = \sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix}_{q_i} f_i^{r-s} f_j f_i^s \quad ; \quad \theta^\pm_{ij} \in \tilde{\mathcal{U}}^\pm_{\pm(r\alpha_i + \alpha_j)}$$

Recall $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$ $\quad [n]_q! = [n]_q \cdots [1]_q \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = \dfrac{[n]_q!}{[n-m]_q! \, [m]_q!}$

and $q_i = q^{d_i}$ $(\forall i \in I)$.

Prop. $\forall\ i \neq j \in I$, $\theta_{ij}^{\pm} \in \mathrm{rad}^{\pm}$

Proof. Let us prove it for $+$ case. $-$ case is proved similarly, (or one could use automorphism ω of $\tilde{\mathcal{U}}$: $\omega(e_i) = f_i$, $\omega(f_i) = e_i$, $\omega(h) = -h$).

Recall that we introduced derivations r_i, r_i' of $\tilde{\mathcal{U}}^+$ last time:

$$r_i(e_j) = \delta_{ij} \qquad r_i(xx') = q^{(\mu', \alpha_i)} r_i(x)\, x' + x\, r_i(x')$$

$$r_i'(e_j) = \delta_{ij} \qquad r_i'(xx') = r_i'(x)\, x' + q^{(\mu, \alpha_i)} x\, r_i'(x')$$

It suffices to prove that $r_k(\theta_{ij}^+) = 0 \quad \forall\ k \in I$.

For $k \neq i, j$, $r_k(\theta_{ij}^+) = 0$ is clear.

For $k = j$: $\quad r_j(\theta_{ij}^+) = \left[\sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix}_i q_i^{s a_{ij}}\right] e_i^r = \left[\sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix}_i q_i^{(1-r)s}\right] e_i^r$

Use the identity $\begin{bmatrix} n+1 \\ m \end{bmatrix} = q^m \begin{bmatrix} n \\ m \end{bmatrix} + q^{m-n-1} \begin{bmatrix} n \\ m-1 \end{bmatrix}$ and prove by induction on r that $\sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix} q^{(1-r)s} = 0$.

For $k = i$:

$$r_i(e_i^n e_j e_i^m) = q_i^{2m+n-1+a_{ij}} [n]_i\, e_i^{n-1} e_j e_i^m + q_i^{m-1} [m]_i\, e_i^n e_j e_i^{m-1}$$

Thus coeff of $e_i^{r-1-a} e_j e_i^a$ in $r_i(\theta_{ij}^+)$ is given by $(0 \le a \le r-1)$

$$(-1)^a \begin{bmatrix} r \\ a \end{bmatrix}_i q_i^{2a+r-a-1+a_{ij}} [r-a]_i + (-1)^{a+1} \begin{bmatrix} r \\ a+1 \end{bmatrix} q_i^a [a+1]_i$$

$$= (-1)^a \frac{[r]!}{[r-a-1]!\,[a]!} \left(q_i^a - q_i^a \right) = 0$$

$\square$

Remark: in fact $\mathrm{rad}^{\pm} = \langle \theta_{ij}^{\pm} : i \neq j \rangle$, but we won't need it so I will not prove it.

(5.3) Representations of $U_q\mathfrak{g}$.

A representation V of $U_q\mathfrak{g}$ is called $\mathfrak{h}$-diagonalizable if $V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu]$

$$V[\mu] = \{ v \in V \mid h.v = \mu(h) v \ \forall h \in \mathfrak{h} \}$$

Category $\mathcal{O}$: An $\mathfrak{h}$-diagonalizable repn. of $U_q\mathfrak{g}$ is in category $\mathcal{O}$ if $\exists \lambda_1, \dots, \lambda_r \in \mathfrak{h}^*$ s.t. $V[\mu] \neq 0 \Rightarrow \mu \leq \lambda_i$ for some $i = 1 \dots r$.

$P(V) := \{ \mu \in \mathfrak{h}^* \mid V[\mu] \neq 0 \}$ weights of V. Then the condition above can be written as: $P(V) \subset \bigcup_{i=1\dots r} \lambda_i - Q_+$

Verma Modules: $\forall \lambda \in \mathfrak{h}^*$ define $M_\lambda \in \mathcal{O}$ as repn. gen. by $\mathbb{1}_\lambda$ s.t.

$$e_i \mathbb{1}_\lambda = 0 \ (\forall i \in I) \qquad h. \mathbb{1}_\lambda = \lambda(h) \mathbb{1}_\lambda.$$

In other words M_λ is quotient of $U_q\mathfrak{g}$ by left ideal generated by e_i $(i \in I)$ and $h - \lambda(h)$ $(h \in \mathfrak{h})$. It is the universal highest weight representation of highest weight λ. That is, if $V \in \mathcal{O}$ and $v \in V[\mu]$ is a primitive vector (i.e. $e_i v = 0 \ \forall i \in I$) then $\exists!$ $U_q\mathfrak{g}$-module homomorphism

$$M_\mu \to V$$
$$\mathbb{1}_\mu \mapsto v$$

Any proper submodule of M_λ avoids the highest weight vector. Hence $\exists!$ max'l proper submodule I_λ of M_λ. Define $L_\lambda = M_\lambda / I_\lambda$

Clearly L_λ is irreducible. Conversely if V is an irr. repn $\in \mathcal{O}$, choose $\mu \in P(V)$ to be a max'l element. Then $e_i v = 0 \ \forall i \in I$ $v \in V[\mu]$.

$\Rightarrow M_\mu \twoheadrightarrow V$ and hence $L_\mu \simeq V$ (by irreducibility).

Remark: Category $\mathcal{O}$ is a tensor category. Moreover for $V, W \in \mathcal{O}$

$R_{V,W} = \pi_V \otimes \pi_W (R)$ is well defined $(\pi_V, \pi_W : U_q\mathfrak{g} \to End(V), End(W))$

element of $End(V \otimes W)$. Thus $\mathcal{O}$ is braided by commutativity

constraint $\quad c_{V,W} = \sigma \circ R_{V,W}$.

Lemma $\quad C_q$ acts as $q^{(\lambda+2\rho, \lambda)} \cdot Id$ on M_λ.

Proof $\quad C_q = q^{2\rho} u^{-1}$ and $u^{-1} = \mu \circ S^{-1} \otimes S (R_{21})$. Then u^{-1} is

of the form $\left(\sum S^{-1}(b_l) S(a_l) \right) \cdot q^{\sum x_i \otimes x^i}$

On 1_λ, $\sum x_i x^i$ acts by $\sum \lambda(x_i) \lambda(x^i) = (\lambda, \lambda)$

$S(a_l) \in U^+ \Rightarrow S(a_l) 1_\lambda = 0$ unless $a_l = 1$.

$\Rightarrow \quad C_q 1_\lambda = q^{(\lambda, 2\rho)} q^{(\lambda,\lambda)} 1_\lambda = q^{(\lambda+2\rho, \lambda)} 1_\lambda$. Finally

M_λ is generated by 1_λ and C_q is central. $\square$

Example of sl_2 $\quad$ Let $\lambda \in \mathbb{C}$. Then M_λ has the form

$$M_\lambda = \text{span of } \{ 1_\lambda, f 1_\lambda, f^{(2)} 1_\lambda, \ldots \}$$

Notation $\quad f^{(r)} = \dfrac{f^r}{[r]!}$. $\quad U_q sl_2$ action on M_λ is given by

$$f \cdot (f^{(r)} 1_\lambda) = [r+1] (f^{(r+1)} 1_\lambda) \qquad h. f^{(r)} 1_\lambda = (\lambda - 2r) f^{(r)} 1_\lambda$$

Now we have the following commutation relation

$$e f^{(r)} = f^{(r)} e + \frac{q^{r-1} k - q^{-r+1} k^{-1}}{q - q^{-1}} f^{(r-1)}$$

$\Rightarrow \quad e.\left(f^{(r)}\mathbb{1}_\lambda\right) = [\lambda-r+1]\, f^{(r-1)}\mathbb{1}_\lambda$

If $\lambda \in \mathbb{N}$, M_λ has a submodule generated by $f^{(\lambda+1)}\mathbb{1}_\lambda$. otherwise M_λ is irreducible. Thus for $\lambda \in \mathbb{N}$ we get simple L_λ of dimension $\lambda+1$.

(5.4) Integrable representations. $V \in \mathcal{O}$ is said to be integrable if each f_i acts locally nilpotently on V.

$\mathcal{O}_{int}$ = subcategory of integrable representations (extension closed sub/quotient/finite $\oplus$) (again a braided tensor category).

Theorem. Simple (or irreducible) objects of $\mathcal{O}_{int}$ are L_λ's where $\lambda \in P_+$ (i.e. $(\lambda,\alpha_i) \in \mathbb{N}$ $\forall i$). $\mathcal{O}_{int}$ is a semisimple category.

Proof. We prove this theorem in following steps:

(a) L_λ is integrable iff $\lambda \in P_+$. In this case

$$L_\lambda = M_\lambda \Big/ \left\langle f_i^{\lambda(h_i)+1}.\mathbb{1}_\lambda \right\rangle$$

(b) $V \in \mathcal{O}_{int}$ is irr. $\Rightarrow$ $V \simeq L_\lambda$ for some $\lambda \in P_+$

(c) $V \in \mathcal{O}_{int} \Rightarrow V$ is sum of simple objects (and hence a direct sum).

(a) Let $\lambda \in P_+$. Then J_λ = submodule gen. by $f_i^{\lambda(h_i)+1}\mathbb{1}_\lambda$ $(i \in I)$ is a proper submodule

since $e_i f_i^{(r)}\mathbb{1}_\lambda = [\lambda(h_i)-r+1]_{q_i}\, f_i^{(r-1)}\mathbb{1}_\lambda$

$$\tilde{L}_\lambda = M_\lambda / J_\lambda \twoheadrightarrow L_\lambda.$$

Claim. $\tilde{L}_\lambda$ is integrable. This is clear since by Serre relations each product $f_i^N f_j$ can be written in terms of $f_i^a f_j f_i^{N-a}$ with $0 \le a < r$ $(r = 1 - a_{ij})$.

Hence L_λ is integrable being a quotient of $\tilde{L}_\lambda$. Moreover if $\tilde{L}_\lambda \neq L_\lambda$ there will exist $v \in \tilde{L}_\lambda[\lambda']$ s.t. $e_i v = 0$ $\forall i \in I$; $\lambda' \le \lambda$, $\lambda' \in P_+$.

But then using q-Casimir element $\boxed{(\lambda' + 2\rho, \lambda') = (\lambda + 2\rho, \lambda) \Rightarrow \lambda = \lambda'}^*$

Now assume $\lambda \in \mathfrak{h}^*$ is s.t. L_λ is integrable. $\forall i \in I$ $\exists N_i$ s.t. $f_i^{N_i+1} \mathbb{1}_\lambda = 0$, $f_i^{N_i} \mathbb{1}_\lambda \neq 0$. Again we get

$$0 = e_i f_i^{(N_i+1)} \mathbb{1}_\lambda = [\lambda(h_i) - N_i]_i f_i^{(N_i)} \mathbb{1}_\lambda \Rightarrow \lambda(h_i) = N_i \in \mathbb{N}.$$

(b) has already been proved.

(c) Let $V \in \mathcal{O}_{int}$. $V^0 = \{v \in V \mid e_i v = 0 \ \forall i\} = \bigoplus_{\lambda \in \mathfrak{h}^*} V^0[\lambda]$

each $\lambda \in P_+$ and set V_1 = submodule gen. by $v \in V^0$ $(\forall v \in V^0)$

Then V_1 = sum of simple modules and hence a direct sum.

We claim $V_1 = V$. If not $V_2 := V/V_1$ is again integrable and hence contains a primitive vector $v \in V_2[\lambda']$ $\lambda' \in P_+$.

$e_i v \in V_1 \Rightarrow \lambda' \le \lambda$ for some $\lambda \in P_+$ s.t. $V^0[\lambda] \neq 0$.

q-Casimir $\Rightarrow (\lambda' + 2\rho, \lambda') = (\lambda + 2\rho, \lambda) \Rightarrow \lambda = \lambda'$ contradiction. $\square$

Proof of $(*)$: $\lambda, \lambda' \in P_+$, $\lambda \ge \lambda'$, $(\lambda + 2\rho, \lambda) = (\lambda' + 2\rho, \lambda') \Rightarrow \lambda = \lambda'$.

$\lambda - \lambda' = \sum_{i \in I} n_i \alpha_i$, $n_i \ge 0$. Now $(\lambda, \lambda) - (\lambda', \lambda') \ge (\lambda, \lambda - \lambda') = \sum n_i (\lambda, \alpha_i)$

$$\Rightarrow 0 = (\lambda + 2\rho, \lambda) - (\lambda' + 2\rho, \lambda') \ge \sum n_i (\lambda + 2\rho, \alpha_i)$$

$$\Rightarrow n_i = 0 \ \forall i \qquad \square$$