(6.0) Recall: $\mathfrak{g}$ is an arbitrary (symmetrizable) Kac-Moody algebra

$U_q\mathfrak{g}$ is the associated quantum group.

Category $\mathcal{O}$. A representation V of $U_q\mathfrak{g}$ is in category $\mathcal{O}$ if

1. $V = \bigoplus_{\mu\in\mathfrak{h}^*} V[\mu]$ $\quad \dim V[\mu] < \infty$ $\quad$ ($\mathfrak{h}$-diagonalizable)

2. $\exists\ \lambda_1, \dots, \lambda_r \in \mathfrak{h}^*$ s.t. $V[\mu] \neq 0 \Rightarrow \mu \leq \lambda_i$ for some i

Integrable representations. A representation V of $U_q\mathfrak{g}$ is integrable if it is $\mathfrak{h}$-diagonalizable and each e_i, f_i act locally nilpotently.

Let $B_\mathfrak{g}$ be the braid group associated to $\mathfrak{g}$, i.e.

$$B_\mathfrak{g} = \langle T_i \ (i\in I) \mid \underbrace{T_iT_jT_i\cdots = T_jT_iT_j\cdots}_{m_{ij}\text{ factors on both sides}} \rangle$$

$m_{ij} = 2, 3, 4, 6, \infty$ if $a_{ij}a_{ji} = 0, 1, 2, 3, \geq 4$ respectively.

Aim: to define $B_\mathfrak{g}$-action on each integrable $U_q\mathfrak{g}$ representation.

(6.1) Let us assume $\mathfrak{g} = \mathfrak{sl}_2$ for time being. Define

$$S = \exp_{q^{-1}}(q^{-1}ek^{-1})\,\exp_{q^{-1}}(-f)\,\exp_{q^{-1}}(qek)\,q^{\frac{h(h+1)}{2}}$$

where $\exp_q(x) = \sum_{n\geq 0} q^{n(n-1)/2}\dfrac{x^n}{[n]!}$

Note • $\exp_q(x)^{-1} = \exp_{q^{-1}}(-x) \Rightarrow S$ is an invertible operator.

• S makes sense on each integrable $U_q\mathfrak{sl}_2$-representation.

Lemma. Let $\lambda \in \mathbb{N}$ and L_λ be the irreducible $(\lambda+1)$-dim'l repn. of $U_q sl_2$: basis of $L_\lambda = \{ m_\lambda(r) : r = 0, \dots, \lambda \}$

$U_q sl_2$-action

$$h \, m_\lambda(r) = (\lambda - 2r) \, m_\lambda(r)$$

$$f \, m_\lambda(r) = [r+1] \, m_\lambda(r+1) \qquad \left(m_\lambda(\lambda+1) = m_\lambda(-1) = 0 \right)$$

$$e \, m_\lambda(r) = [\lambda - r + 1] \, m_\lambda(r-1)$$

Then

$$\mathcal{S} \, m_\lambda(r) = (-1)^{\lambda - r} q^{(\lambda - r)(r+1)} m_\lambda(\lambda - r) = \sum_{\substack{a,b,c \geq 0 \\ b-a-c = \lambda - 2r}} (-1)^b q^{b-ac} e^{(a)} f^{(b)} e^{(c)} m_\lambda(r)$$

Proof. Let us recall the definition of q-binomial coeff

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m][m-1] \cdots [m-n+1]}{[n]!} \qquad \forall \, m \in \mathbb{Z}, \ n \in \mathbb{N}.$$

Then we have $\begin{bmatrix} m \\ n \end{bmatrix} = (-1)^n \begin{bmatrix} n-1-m \\ n \end{bmatrix}$. Moreover we have

$$\sum_{i=0}^{n} q^{ai - b(n-i)} \begin{bmatrix} a \\ n-i \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix} = \begin{bmatrix} a+b \\ n \end{bmatrix} \qquad \forall \, a, b \in \mathbb{Z}, \ n \in \mathbb{N}$$

From the definitions coefficient of $m_\lambda(\lambda - t)$ in $\mathcal{S} \, m_\lambda(r)$ is given by

$$q^{\frac{(\lambda-2r)(\lambda-2r+1)}{2}} \sum_{\substack{a,b,c \geq 0 \\ a-b+c = r+t-\lambda}} (-1)^b q^{\frac{a(a+1)}{2} - \frac{b(b-1)}{2} + \frac{c(c+1)}{2} + c(\lambda-2r) + a(\lambda-2t)} \begin{bmatrix} \lambda-r+c \\ c \end{bmatrix} \begin{bmatrix} r-c+b \\ b \end{bmatrix} \begin{bmatrix} t \\ a \end{bmatrix}$$

$$= \sum_a (-1)^{\lambda - r + a - t} \begin{bmatrix} t \\ a \end{bmatrix} q^{a(r-t+1) + \lambda(t+1) - rt + \frac{r(r-1)}{2} - \frac{t(t+1)}{2}}$$

$$\sum_c (-1)^c q^{(r-c)(r-\lambda-1) - c(\lambda - t + a)} \begin{bmatrix} \lambda-r+c \\ c \end{bmatrix} \begin{bmatrix} \lambda-t+a \\ r-c \end{bmatrix}$$

$$\left(\text{since} \ \begin{bmatrix} r-c+b \\ b \end{bmatrix} = \begin{bmatrix} r-c+b \\ r-c \end{bmatrix} = \begin{bmatrix} \lambda-t+a \\ r-c \end{bmatrix} \right)$$

$$= \sum_a (-1)^{\lambda-r+a-t} \begin{bmatrix} t \\ a \end{bmatrix} q^{a(r-t+1)+\lambda(t+1)-rt+\frac{r(r-1)}{2}-t\frac{(t+1)}{2}} \cdot (-1)^r \begin{bmatrix} t-a \\ r \end{bmatrix}$$

Now $(-1)^r \begin{bmatrix} t-a \\ r \end{bmatrix} = (-1)^{t-a} \begin{bmatrix} -r-1 \\ t-a-r \end{bmatrix}$

$$= \sum_a (-1)^{\lambda-r} q^{\lambda(t+1)-rt+\frac{r(r-1)}{2}-\frac{t(t+1)}{2}-t(t-r)} q^{t(t-r-a)+a(r+1)} \begin{bmatrix} t \\ a \end{bmatrix} \begin{bmatrix} -r-1 \\ t-a-r \end{bmatrix}$$

$$= (-1)^{\lambda-r} q^{\lambda(t+1)-rt+\frac{r(r-1)}{2}-\frac{t(t+1)}{2}-t(t-r)} \begin{bmatrix} t-r-1 \\ t-r \end{bmatrix}$$

$$= \delta_{t,r} (-1)^{\lambda-r} q^{\lambda(r+1)-r^2-r} = \delta_{t,r} (-1)^{\lambda-r} q^{(\lambda-r)(r+1)} \qquad \square$$

Cor. $\mathrm{Ad}(\mathbb{S})\, e = -fk \qquad \mathrm{Ad}(\mathbb{S}) f = -k^{-1} e \qquad \mathrm{Ad}(\mathbb{S})\, h = -h$

Proof. $\mathbb{S}^{-1} m_\lambda(t) = (-1)^t q^{-t(\lambda-t+1)} m_\lambda(\lambda-t)$

$$\Rightarrow \quad \mathbb{S}\, e\, \mathbb{S}^{-1} m_\lambda(r) = \left(\mathbb{S}\, e\, m_\lambda(\lambda-r)\right)(-1)^r q^{-r(\lambda-r+1)}$$

$$= (-1)^r q^{-r(\lambda-r+1)} [r+1]\, \mathbb{S}\, m_\lambda(\lambda-r-1)$$

$$= (-1)^{r+r+1} q^{-r(\lambda-r+1)+(r+1)(\lambda-r)} [r+1]\, m_\lambda(r+1)$$

$$= -q^{\lambda-2r} [r+1]\, m_\lambda(r+1) = -fk\, m_\lambda(r). \qquad \square$$

Let us denote the automorphism defined in Corollary above by T

$$T(e) = -fk \qquad T(f) = -k^{-1} e \qquad T(h) = -h$$

Remark. In order to make the statement of the Corollary above more precise we can say that $\forall\, u \in U_q\mathfrak{sl}_2 \ \exists!\ T(u) \in U_q\mathfrak{sl}_2$ s.t. $\mathbb{S}.(u(v)) = T(u).\,\mathbb{S} v \quad \forall\, v \in L \quad \forall\, L \in$ Integrable repns. (or $\mathcal{O}_{int}$)

We will need the following "separation of points" argument to ensure uniqueness and the fact that T is an algebra automorphism.

Separation of points: if $u \in U_q sl_2$ acts as 0 on each integrable $U_q sl_2$-repn. then $u = 0$.

(6.2) Relation with coproduct.

Thm. $\Delta(\mathcal{S}) = \mathcal{S} \otimes \mathcal{S} \cdot \bar{R}$ where

$$\bar{R} = \exp_q\left((q - q^{-1})\, e \otimes f\right) = q^{-\frac{h \otimes h}{2}} R$$

Proof. We first check that both $\Delta(\mathcal{S})$ and $\mathcal{S} \otimes \mathcal{S} \cdot \bar{R}$ have same commutation relation with $\Delta(x)$, $x \in U_q sl_2$. In fact it is enough to prove this assertion for $x = e, f$ and h.

$x = e$.
$$\Delta(\mathcal{S})\, \Delta(e)\, \Delta(\mathcal{S})^{-1} = \Delta(Te) = \Delta(-fk)$$
$$= -(f \otimes 1 + k^{-1} \otimes f)(k \otimes k) = -fk \otimes k - 1 \otimes fk.$$
$$= \mathrm{Ad}(\mathcal{S} \otimes \mathcal{S})\, \mathrm{Ad}(q^{-\frac{h \otimes h}{2}})\, \Delta^{21}(e)$$

$$\mathrm{Ad}(\mathcal{S} \otimes \mathcal{S})\, \mathrm{Ad}(q^{-h \otimes h/2})\, \mathrm{Ad}(R) \cdot \Delta(e) =$$
$$\mathrm{Ad}(\mathcal{S} \otimes \mathcal{S})\, \mathrm{Ad}(q^{-h \otimes h/2})\, (k \otimes e + e \otimes 1)$$
$$= \mathrm{Ad}(\mathcal{S} \otimes \mathcal{S})\, \mathrm{Ad}(q^{-h \otimes h/2}) = -1 \otimes fk - fk^{+1} \otimes k$$
$$= \mathrm{Ad}(\mathcal{S} \otimes \mathcal{S})\, (1 \otimes e + e \otimes k^{-1})$$

Similarly for $x = f$ and h.

To prove this equation for $L_\lambda \otimes L_\mu$ $(\lambda, \mu \in \mathbb{N})$ it is now enough to check it on a cyclic vector. Take $\xi = m_\lambda(0) \otimes m_\mu(\mu)$ (highest $\otimes$ lowest)

It is clear that ξ is cyclic i.e. $U_q sl_2 \cdot \xi = L_\lambda \otimes L_\mu$:

$$\sum_n \mathbb{C} e^n \xi = \mathbb{C}\, m_\lambda(0) \otimes L_\mu$$ apply lowering operators to get everything.

(5)

$$\mathcal{S}\otimes\mathcal{S}\ \overline{R}\cdot\xi = \mathcal{S}\, m_\lambda(0)\otimes \mathcal{S}\, m_\mu(\mu)$$
$$= (-1)^\lambda q^\lambda\ m_\lambda(\lambda)\otimes m_\mu(0).$$

It remains to show that $\Delta(\mathcal{S})\cdot\xi = (-1)^\lambda q^\lambda m_\lambda(\lambda)\otimes m_\mu(0)$

We will need the following coproduct identities $\qquad e^{(a)} = \dfrac{e^a}{[a]!}$

$$\Delta(e^{(n)}) = \sum_{i=0}^{n} q^{i(n-i)}\, e^{(n-i)}\otimes e^{(i)} k^{n-i}$$
$$\Delta(f^{(n)}) = \sum_{i=0}^{n} q^{i(n-i)}\, f^{(i)} k^{-(n-i)}\otimes f^{(n-i)}$$

$$\mathcal{S}\, m_\lambda(0)\otimes m_\mu(\mu) = \sum_{\substack{a,b,c\geq 0\\ b-a-c=\lambda-\mu}} (-1)^b q^{b-ac}\, e^{(a)} f^{(b)} e^{(c)}\cdot (m_\lambda(0)\otimes m_\mu(\mu))$$

$$= \sum_{\substack{a,b,c\geq 0\\ b-a-c=\lambda-\mu}} (-1)^b q^{b-ac}\, e^{(a)} f^{(b)} \left(m_\lambda(0)\otimes m_\mu(\mu-c)\right)$$

$$= \sum_{b'+b''=b} (-1)^{b'+b''} q^{b'+b''-ac+b'b''-b''\lambda}\, e^{(a)}\cdot\left(m_\lambda(b')\otimes \begin{bmatrix}\mu-c+b''\\ b''\end{bmatrix} m_\mu(\mu-c+b'')\right)$$

$$= \sum_{a'+a''=a} (-1)^{b'+b''} q^{b'+b''-ac+b'b''-b''\lambda+a'a''+a'(-\mu+2c-2b'')} \begin{bmatrix}\mu-c+b''\\ b''\end{bmatrix}\begin{bmatrix}\lambda-b'+a'\\ a'\end{bmatrix}\begin{bmatrix}c-b''+a''\\ a''\end{bmatrix}$$
$$\cdot\, m_\lambda(b'-a')\otimes m_\mu(\mu-c+b''-a'')$$

Set $k = b'-a'$

$$= \sum (-1)^{g+b''} q^{g+b''-a''c+b''g-b''\lambda}\begin{bmatrix}\mu-c+b''\\ b''\end{bmatrix}\begin{bmatrix}c-b''+a''\\ a''\end{bmatrix}\cdot \underbrace{\sum (-1)^{a'} q^{a'(1-\lambda+g)}\begin{bmatrix}\lambda-g\\ a'\end{bmatrix}}_{=0 \text{ unless } \lambda=g}$$
$$\cdot\, m_\lambda(g)\otimes m_\mu(\mu-c+b''-a'')$$

Thus $\lambda = g$ and we have

$$b'' = a''+c-\mu \qquad b' = \lambda+a'$$

$$= \sum (-1)^{\lambda-\mu+c+a''} q^{\lambda-\mu+c+a''-a''c}\begin{bmatrix}a''\\ \mu-c\end{bmatrix}\begin{bmatrix}\mu\\ a''\end{bmatrix}\cdot m_\lambda(\lambda)\otimes m_\mu(0)$$

$$\underset{d=\mu-c}{=} \sum (-1)^{\lambda+a''} q^{\lambda+a''-a''\mu}\Big(\underbrace{\sum (-1)^d q^{-d(1-a'')}\begin{bmatrix}a''\\ d\end{bmatrix}}_{=0 \text{ unless } a''=0}\Big)\begin{bmatrix}\mu\\ a''\end{bmatrix} = (-1)^\lambda q^\lambda\, m_\lambda(\lambda)\otimes m_\mu(0)$$

$\square$

(6.3) For arbitrary $\mathfrak{g}$ define for each $i \in I$

$$\mathcal{S}_i = \exp_{q_i^{-1}}(q_i^{-1} e_i k_i^{-1}) \exp_{q_i^{-1}}(-f_i) \exp_{q_i^{-1}}(q_i e_i k_i)\, q_i^{h_i(h_i+1)/2}$$

Thus if V is an integrable representation of $U_q\mathfrak{g}$ and $v \in V[\mu]$

$$\mathcal{S}_i v = \sum_{\substack{a,b,c \geq 0 \\ b-a-c = \mu(h_i)}} (-1)^b q_i^{b-ac} e_i^{(a)} f_i^{(b)} e_i^{(c)} v \quad \in V[s_i\mu]$$

Again let $T_i = \mathrm{Ad}(\mathcal{S}_i)$

Prop. $T_i(e_i) = -f_i k_i \qquad T_i(f_i) = -k_i^{-1} e_i \qquad T_i(h) = s_i(h)$

$$\forall j \neq i \qquad T_i(e_j) = \sum_{s=0}^{r} (-1)^s q_i^{-s} e_i^{(r-s)} e_j e_i^{(s)}$$

$$T_i(f_j) = \sum_{s=0}^{r} (-1)^s q_i^{s} f_i^{(s)} f_j f_i^{(r-s)} \qquad r = -a_{ij}$$

Proof: We have already proved the first set of relations. It only remains to prove that $T_i(e_j)$ and $T_i(f_j)$ are given by equations above. Let us prove it for the f case.

Define $U_{q_i}sl_2$ action on $U_q\mathfrak{g}$ via $\varphi : U_q sl_2 \to \mathrm{End}(U_q\mathfrak{g})$:

$$\varphi(e_i)\cdot x = k_i^{-1}[x, e_i]$$

$$\varphi(f_i)\cdot x = x f_i - f_i k_i x k_i^{-1}$$

$$\varphi(h_i)\cdot x = [h_i, x]$$

Then $$f_{ij;\ell} := \varphi(f_i^{(\ell)}).f_j = \sum_{k=0}^{\ell} (-1)^{\ell-k} q_i^{(\ell-k)(1-a_{ij}-\ell)} f_i^{(\ell-k)} f_j f_i^{(k)}$$

In particular Serre relations imply that span of $\{f_{ij;\ell}\}_{\ell \geq 0}$ is irr. $U_q sl_2$ repn. of highest weight $r = -a_{ij}$

$$\mathbb{S}_i = \underbrace{\exp_{\bar{q}_i^{-1}}(\bar{q}_i^{-1} e_i k_i^{-1})}_{A} \underbrace{\exp_{\bar{q}_i^{-1}}(-f_i)}_{B} \underbrace{\exp_{\bar{q}_i^{-1}}(q_i e_i k_i)}_{C} q_i^{\frac{h_i(h_i+1)}{2}}$$

(1) $$q_i^{\frac{h_i(h_i+1)}{2}} \left(f_j k_j^{-1} k_i^{a_{ij}}\right) q_i^{-\frac{h_i(h_i+1)}{2}} = q_i^{a_{ij}(1-a_{ij})/2} f_j k_j^{-1}$$

(2) $$[e_i k_i, f_j k_j^{-1}] = 0 \Rightarrow Ad(C)\, f_j k_j^{-1} = f_j k_j^{-1}$$

(3) $$Ad(B)(f_j k_j^{-1}) = \sum_{\ell \geq 0} q_i^{\frac{\ell(\ell-1)}{2} + \ell a_{ij}} f_{ij;\ell}\, k_j^{-1} \quad \text{(easy check)}$$

(4) $$Ad(A)^{-1}(f_{ij;r} k_j^{-1}) = q_i^{a_{ij}(a_{ij}-1)/2} \sum_{\ell \geq 0} q_i^{\frac{\ell(\ell-1)}{2} + \ell a_{ij}} f_{ij;\ell}\, k_j^{-1} \quad \text{(use } U_{q_i} sl_2 \text{ -action)}$$

Combining, we get $T_i(f_j k_j^{-1} k_i^{a_{ij}}) = f_{ij;r}\, k_j^{-1}$

$\Rightarrow T_i(f_j) = f_{ij;r}$ as claimed $\square$

(6.4) Interesting application. $\text{rad}^- = \langle \theta_{ij}^- = f_{ij;1-a_{ij}} : i \neq j \in I \rangle =: I^-$

We have already proved that $I^- \subset \text{rad}^-$. Note that only Serre relations were used in obtaining T_i's.

Assume on the contrary that $\text{rad}^- \supsetneq I^-$. Let $r \in \text{rad}^-_{-\mu} \setminus I^-_{-\mu}$ be a nontrivial element with minimal μ. Since T_i preserve the radical we get $s_i(\mu) > \mu \Rightarrow (\mu, \alpha_i) \leq 0$.

Next we take the projection $\pi : \tilde{\mathcal{U}} \to \mathcal{U}$ and define $\mathcal{U}$-mod homomorphism

$$\psi : \text{rad}^- / (\text{rad}^-)^2 \longrightarrow \bigoplus_{i \in I} M_{-\alpha_i}$$

$$\sum u_i f_i \longmapsto \sum \pi(u_i) \cdot \mathbb{1}_{-\alpha_i}$$

Since r is min'l $[e_i, r] = 0 \quad \forall i$

$\Rightarrow \quad \psi(r)$ is a primitive vector. Using q-Casimir element

we get $\quad (-\mu+2\rho, -\mu) = (-\alpha_i+2\rho, -\alpha_i) = 0$

$\Rightarrow \quad (\mu, \mu) = (\mu, 2\rho)$. This yields a contradiction since

if $\quad \mu = \sum_{i\in I} n_i \alpha_i \quad (n_i \geq 0)$

$(\mu, 2\rho) = \sum_{i\in I} n_i(\alpha_i, \alpha_i) \geq 0 \qquad (\mu,\mu) = \sum n_i(\mu, \alpha_i) \leq 0 \Rightarrow n_i = 0 \ (\forall i)$.

(6.5) Thm. $\{\mathcal{S}_i\}_{i\in I}$ satisfy braid relations.

The proof of this theorem is computational and case by case.
The proof of this theorem is computational and case by case.
the braid relations in types

Namely, we have to verify

$A_1 \times A_1 \ : \ a_{ij} = a_{ji} = 0 \qquad \mathcal{S}_i \mathcal{S}_j = \mathcal{S}_j \mathcal{S}_i$

$A_2 \ : \ a_{ij} = a_{ji} = -1 \qquad \mathcal{S}_i \mathcal{S}_j \mathcal{S}_i = \mathcal{S}_j \mathcal{S}_i \mathcal{S}_j$

$B_2 \ : \ a_{ij} = -2 \quad a_{ji} = -1 \qquad \mathcal{S}_i \mathcal{S}_j \mathcal{S}_i \mathcal{S}_j = \mathcal{S}_j \mathcal{S}_i \mathcal{S}_j \mathcal{S}_i$

$G_2 \ : \ a_{ij} = -3 \quad a_{ji} = -1 \qquad \mathcal{S}_i \mathcal{S}_j \mathcal{S}_i \mathcal{S}_j \mathcal{S}_i \mathcal{S}_j = \mathcal{S}_j \mathcal{S}_i \mathcal{S}_j \mathcal{S}_i \mathcal{S}_j \mathcal{S}_i$

Below we prove it in the first two cases.

Proof for $A_1 \times A_1$: To show: $\mathcal{S}_i \mathcal{S}_j \mathcal{S}_i^{-1} = \mathcal{S}_j \equiv T_i(\mathcal{S}_j) = \mathcal{S}_j$

This is clear since $\quad T_i(e_j) = e_j \quad T_i(f_j) = f_j \quad T_i(h_j) = h_j$

$(a_{ij} = a_{ji} = 0)$

Proof for A_2: we have $T_i(e_j) = e_i e_j - q^{-1} e_j e_i$

$$T_i(e_i) = -f_i k_i$$

Claim $T_1T_2(e_1) = e_2$ $\quad T_1T_2(f_1) = f_2$ $\quad T_1T_2(h_1) = h_2$

Let us verify it for e's.

$$T_2(e_1) = e_2e_1 - q^{-1}e_1e_2$$

$$\begin{aligned}
T_1T_2(e_1) &= (e_1e_2 - q^{-1}e_2e_1)(-f_1k_1) - q^{-1}(-f_1k_1)(e_1e_2 - q^{-1}e_2e_1) \\
&= -(e_1e_2 - q^{-1}e_2e_1)(f_1k_1) + (f_1e_1e_2 - q^{-1}f_1e_2e_1)k_1 \\
&= -(e_1e_2 - q^{-1}e_2e_1)(f_1k_1) + (e_1e_2 - q^{-1}e_2e_1)f_1k_1 \\
&\qquad + \left(-\frac{k_1-k_1^{-1}}{q-q^{-1}}e_2 + q^{-1}e_2\frac{k_1-k_1^{-1}}{q-q^{-1}}\right)k_1 \\
&= \frac{1}{q-q^{-1}}\begin{bmatrix} -q^{-1}e_2k_1 + q\,e_2k_1^{-1} \\ + q^{-1}e_2k_1 - q^{-1}e_2k_1^{-1} \end{bmatrix} k_1 = e_2.
\end{aligned}$$

Hence $\quad \mathcal{S}_1\mathcal{S}_2\mathcal{S}_1\mathcal{S}_2^{-1}\mathcal{S}_1^{-1} = T_1T_2(\mathcal{S}_1) = \mathcal{S}_2 \qquad \square$