Lecture 7 Fusion operator and Dynamical Weyl group

(7.0) $\mathfrak{g}$ = Kac-Moody algebra $U_q\mathfrak{g}$ = quantum group

Let $\lambda \in \mathfrak{h}^*$ and M_λ be the Verma module. For $V \in \mathcal{O}_{int}$ define the expectation value map as

$$\mathrm{Hom}_{\mathcal{U}}(M_\lambda, M_\mu \otimes V) \longrightarrow V[\lambda-\mu]$$
$$\varphi \longmapsto \langle\varphi\rangle \quad \text{where } \varphi(\mathbb{1}_\lambda) = \mathbb{1}_\mu \otimes \langle\varphi\rangle + \cdots$$

Prop (a) M_λ is generically irreducible. More precisely, if $(\lambda+\rho, \beta) \neq \frac{(\beta,\beta)}{2}$ $\forall \beta \in Q_+$, then M_λ is irreducible.

(b) Let $V \in \mathcal{O}_{int}$ and let γ be a weight of V. Then

$$\mathrm{Hom}_{\mathcal{U}}(M_\lambda, M_{\lambda-\gamma} \otimes V) \to V[\gamma]$$

is an iso. for generic λ or $\lambda \in P_+$, $\lambda \gg \gamma$.

Proof. (a) M_λ reducible $\Rightarrow \exists \beta > 0$ s.t. $M_\lambda[\lambda-\beta]$ contains a primitive vector $\Rightarrow (\lambda+2\rho, \lambda) = (\lambda-\beta+2\rho, \lambda-\beta) \Rightarrow (\lambda+\rho, \beta) = \frac{(\beta,\beta)}{2}$.

(b) $\mathrm{Hom}_{\mathcal{U}}(M_\lambda, M_{\lambda-\gamma} \otimes V)$ = Space of primitive weight λ vectors in $M_{\lambda-\gamma} \otimes V$ $\mathcal{U}^-$

$$= \left[\bigoplus_{\substack{\mu \in Q_+ \\ \mu+\gamma \in P(V)}} M_{\lambda-\gamma}[\lambda-\gamma-\mu] \otimes V[\gamma+\mu]\right]$$

The latter is a finite direct sum by assumptions on V. Hence for λ s.t. $M_{\lambda-\gamma}[\lambda-\gamma-\mu]$ does not contain any primitive vectors (for $\mu \neq 0$), projection onto $\mu = 0$ summand is an iso.

(7.1) For $V \in \mathcal{O}_{int}$, $v \in V[\mu]$ and $\lambda \in \mathfrak{h}^*$ generic, let

$\varphi_\lambda^v : M_\lambda \to M_{\lambda-\mu} \otimes V$ s.t. $\langle \varphi_\lambda^v \rangle = v$. We remark that entries of φ_λ^v are rational functions of q^λ.

e.g. $\mathfrak{g} = sl_2$. Let $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$ and $0 \le k \le n$. Then set $\mu = \lambda - n + 2k$ and we have an intertwiner

$$\varphi(m_\lambda(0)) = \sum_{r=0}^{k} (-1)^r q^{-r(n-2k+r+1)} \frac{\begin{bmatrix} n-k+r \\ r \end{bmatrix}}{\begin{bmatrix} \mu \\ r \end{bmatrix}} m_\mu(r) \otimes m_n(k-r)$$

$\varphi : M_\lambda \to M_\mu \otimes L_n$ s.t. $\langle \varphi \rangle = m_n(k) \in L_n$.

(Proof. Set $\varphi(m_\lambda(0)) = \sum_{r \ge 0} c_r \, m_\mu(r) \otimes m_n(k-r)$. Since $\Delta(e) = e \otimes k + 1 \otimes e$, the r.h.s. is primitive iff

$$0 = \sum_{r \ge 0} c_r \left(q^{n-2k+r} [\mu - r + 1] \, m_\mu(r-1) \otimes m_n(k-r) + [n-k+r+1] \, m_\mu(r) \otimes m_n(k-r-1) \right)$$

This yields recurrence relation ($c_0 = 1$) $c_{r+1} = (-1) q^{-(n-2(k-r-1))} \dfrac{[n-k+r+1]}{[\mu - r]} c_r$)

In particular for $p \ge 0$

$$\varphi(m_\lambda(p)) = \begin{bmatrix} k+p \\ k \end{bmatrix} q^{-p\mu} \, m_\mu(0) \otimes m_n(k+p) + \dots$$

using $\Delta(f^{(p)}) = \sum_{p'+p''=p} q^{p'p''} f^{(p')} k^{-p''} \otimes f^{(p'')}$

Now let $V_1, V_2 \in \mathcal{O}_{int}$ and $v_1 \in V_1[\mu_1]$ $v_2 \in V_2[\mu_2]$.

We define $J_{V_1V_2}(\lambda) \in V_1 \otimes V_2[\mu_1+\mu_2]$ by the following composition

$$M_\lambda \xrightarrow{\varphi_\lambda^{v_2}} M_{\lambda-\mu_2} \otimes V_2 \xrightarrow{\varphi_{\lambda-\mu_2}^{v_1} \otimes \mathrm{Id}_{V_2}} M_{\lambda-\mu_1-\mu_2} \otimes V_1 \otimes V_2$$

Then

$$\varphi_{\lambda-\mu_2}^{v_1} \otimes \mathrm{Id}_{V_2} \circ \varphi_\lambda^{v_2}(\mathbb{1}_\lambda) = \mathbb{1}_{\lambda-\mu_1-\mu_2} \otimes J_{V_1V_2}(\lambda)(v_1 \otimes v_2) + \cdots$$

Prop. (a) $J_{V_1V_2}(\lambda)$ preserves weights; and is lower triangular operator with 1's on the diagonal.

(b) $J_{V_1V_2}(\lambda)$ is a rational function of q^λ.

(Note: (a) means $J_{V_1V_2}(\lambda)(v_1 \otimes v_2) \in v_1 \otimes v_2 + \bigoplus_{\gamma \in Q_+ \setminus \{0\}} V_1[\mu_1-\gamma] \otimes V_2[\mu_2+\gamma]$)

Example of sl_2. $V_1 = L_{n_1}$, $V_2 = L_{n_2}$, $v_1 = m_{n_1}(k_1)$, $v_2 = m_n(k_2)$.

$$J(m_{n_1}(k_1) \otimes m_{n_2}(k_2)) = \sum_{r \geq 0} (-1)^r q^{-r(\lambda-n_1+2k_1+r+1)} \frac{\begin{bmatrix} k_1+r \\ r \end{bmatrix} \begin{bmatrix} n_2-k_2+r \\ r \end{bmatrix}}{\begin{bmatrix} \lambda-n_2+2k_2 \\ r \end{bmatrix}} m_{n_1}(k_1+r) \otimes m_{n_2}(k_2-r)$$

(7.2) Relation with coproduct

Dynamical notation: for $F : \mathfrak{h}^* \to \mathrm{End}(V_1 \otimes \ldots \otimes V_N)$ we write $F(\lambda + h^j)$ for the function ($v_i \in V_i[\mu_i]$)

$$F(\lambda+h^j)(v_1 \otimes \ldots \otimes v_N) = F(\lambda+\mu_j)(v_1 \otimes \ldots \otimes v_N)$$

Prop. Let $V_1, V_2, V_3 \in \mathcal{O}_{int}$. Then

$$J_{V_1 \otimes V_2, V_3}(\lambda)\, J_{V_1,V_2}(\lambda - h^3) = J_{V_1, V_2 \otimes V_3}(\lambda)\, J_{V_2,V_3}(\lambda)$$

Proof. Let $v_i \in V_i[\mu_i]$. Both sides of the equation applied to $v_1 \otimes v_2 \otimes v_3$ are given as follows

$$M_\lambda \longrightarrow M_{\lambda-\mu_3} \otimes V_3 \longrightarrow M_{\lambda-\mu_2-\mu_3} \otimes V_2 \otimes V_3 \longrightarrow M_{\lambda-\mu_1-\mu_2-\mu_3} \otimes V_1 \otimes V_2 \otimes V_3$$

L.H.S. $\cdot$
$$M_{\lambda-\mu_3} \otimes V_3 \longrightarrow M_{\lambda-\mu_2-\mu_3} \otimes V_2 \otimes V_3 \rightarrow M_{\lambda-\mu_1-\mu_2-\mu_3} \otimes V_1 \otimes V_2 \otimes V_3$$

$$1_{\lambda-\mu_3} \otimes v_3 \longmapsto 1_{\lambda-\mu_1-\mu_2-\mu_3} \otimes J_{V_1V_2}(\lambda-\mu_3)(v_1 \otimes v_2) \otimes v_3 + \cdots$$

Compositing with the first map $M_\lambda \rightarrow M_{\lambda-\mu_3} \otimes V_3$ yields

$$1_\lambda \longmapsto 1_{\lambda-\mu_1-\mu_2-\mu_3} \otimes \Big(J_{V_1\otimes V_2, V_3}(\lambda) \big(J_{V_1V_2}(\lambda-\mu_3)(v_1 \otimes v_2) \otimes v_3 \big)\Big) + \cdots$$

Similarly R.H.S.

$$M_\lambda \longrightarrow M_{\lambda-\mu_3} \otimes V_3 \longrightarrow M_{\lambda-\mu_2-\mu_3} \otimes V_2 \otimes V_3$$

$$1_\lambda \longrightarrow 1_{\lambda-\mu_2-\mu_3} \otimes J_{V_2V_3}(\lambda)(v_2 \otimes v_3) + \cdots$$

Composition with $M_{\lambda-\mu_2-\mu_3} \otimes V_2 \otimes V_3 \longrightarrow M_{\lambda-\mu_1-\mu_2-\mu_3} \otimes V_1 \otimes V_2 \otimes V_3$ yields

$$1_\lambda \longrightarrow 1_{\lambda-\mu_1-\mu_2-\mu_3} \otimes J_{V_1, V_2\otimes V_3}(\lambda) \big(v_1 \otimes J_{V_2, V_3}(\lambda)(v_2 \otimes v_3)\big) + \cdots$$ $\square$

(7.3) ABRR equation

Write $R = R_0 \cdot q^{-\sum x_i \otimes x_i}$ $\qquad \theta(\lambda) := \nu^{-1}(\lambda+\rho) - \frac{1}{2}\sum x_i^2$

Thm $J(\lambda)$ is the unique unipotent solution of

$$J(\lambda)\,(1 \otimes q^{2\theta(\lambda)}) = R_0^{21}\,(1 \otimes q^{2\theta(\lambda)})\,J(\lambda)$$

Proof will be given later.

ABRR = Arnaudon Buffenoir Ragoucy Roche

(7.4) Quantum Verma identities.

Let $\lambda \in P_+$, $w = s_{i_1} \cdots s_{i_\ell}$ a reduced expression

$$\alpha^{(j)} := s_{i_\ell} \cdots s_{i_{j+1}} \alpha_{i_j} \quad \text{a positive root} \qquad n_j = 2\,\frac{(\lambda+\rho, \alpha^{(j)})}{(\alpha^{(j)}, \alpha^{(j)})}$$

Lemma. (a) The set $\{(n_j, d_j) : j = 1 \ldots \ell\}$ is independent of the red. exp.

(b) $f_{i_1}^{(n_1)} \cdots f_{i_\ell}^{(n_\ell)} \mathbb{1}_\lambda$ is a singular vector of $M_\lambda[w\cdot\lambda]$ indep. of the reduced expression (denoted by $\mathbb{1}_\lambda(w\cdot\lambda)$) where $w\cdot\lambda = w(\lambda+\rho)-\rho$

Proof. We can check by induction on $l(w)$ that $f_{i_1}^{(n_1)} \cdots f_{i_\ell}^{(n_\ell)} \mathbb{1}_\lambda$ is a singular vector of $M_\lambda[w\cdot\lambda]$. To prove independence from the choice of reduced expression, one only needs to verify it for braid relations. That is, we have to carry out computations in rank 2 $(A_1 \times A_1, A_2, B_2, G_2)$ and $w = w_0$ the longest element. In these cases $\{\alpha^{(j)}\}$ = set of positive roots each with multiplicity 1. (b) reduces to the following $\forall\, a, b \geq 0$

$$A_1 \times A_1 : \quad f_1^a f_2^b = f_2^b f_1^a \qquad A_2 : \quad f_1^a f_2^{a+b} f_1^b = f_2^b f_1^{a+b} f_2^a$$

$$B_2 : \quad f_1^a f_2^{(a+b)} f_1^{a+2b} f_2^b = f_2^b f_1^{a+2b} f_2^{a+b} f_1^a$$

$$G_2 : \quad f_1^a f_2^{a+b} f_1^{2a+3b} f_2^{a+2b} f_1^{a+3b} f_2^b = f_2^b f_1^{a+3b} f_2^{a+2b} f_1^{2a+3b} f_2^{a+b} f_1^a$$

It is possible to check these directly using Serre relations.

More conceptual proof: both sides of the equations give (primitive) singular vectors of $M_\lambda[w_0\cdot\lambda]$. We claim that $\dim M_\lambda[w_0\cdot\lambda]^{U^+} = 1$. If this is the case these vectors will only differ by a scalar which can be checked to be 1 by projecting these onto $\langle f_1, f_2 \mid f_1 f_2 = q_1^{a_{12}} f_2 f_1 \rangle$

$\dim M_\lambda[w_0\cdot\lambda]^{\mathfrak{U}^+} = 1$. Let $k = \dim M_\lambda[w_0\cdot\lambda]^{\mathfrak{U}^+}$. Then each primitive vector will generate a copy of $M_{w_0\cdot\lambda}$ which is irreducible since $\lambda\in P_+$ and w_0 is the longest element.

Hence $M_{w_0\cdot\lambda}^{\oplus k} \hookrightarrow M_\lambda$. Comparing dimensions of weight spaces, we get $k=1$. □

(7.5) Dynamical Weyl group.

Let $w\in W$, $V\in \mathcal{O}_{int}$. Let $\lambda\in P_+$ and consider the intertwining operator for $v\in V[\mu]$ (we assume $\lambda>\mu$)

$$\varphi_\lambda^v : M_\lambda \longrightarrow M_{\lambda-\mu}\otimes V$$

Lemma φ_λ^v maps $M_{w\cdot\lambda}$ to $M_{w\cdot(\lambda-\mu)}\otimes V$

We define $A_w(\lambda)v \in V[w\mu]$ by

$$\varphi_\lambda^v \mathbb{1}_\lambda(w\cdot\lambda) = \mathbb{1}_{\lambda-\mu}(w\cdot(\lambda-\mu))\otimes A_w(\lambda)v + \cdots$$

Proof of Lemma. It suffices to prove it for $\mathfrak{sl}_2$.

$$\varphi_\lambda^v \mathbb{1}_\lambda = \mathbb{1}_{\lambda-\mu}\otimes v + \sum_{p>0} a_p\otimes v_p \; ; \quad a_p\in M_{\lambda-\mu}[\lambda-\mu-p], \; v_p\in V[\mu+p]$$

Now $\mathbb{1}_\lambda(s\cdot\lambda) = f^{(\lambda+1)}\mathbb{1}_\lambda$

$$\Rightarrow \varphi_\lambda^v(\mathbb{1}_\lambda(s\cdot\lambda)) = \underbrace{f^{(m)}\mathbb{1}_{\lambda-\mu}\otimes v'}_{\text{leading term}} + \sum a_p'\otimes v_p'$$

Since $\mathbb{1}_\lambda(s\cdot\lambda)$ is primitive, so must be $f^{(m)}\mathbb{1}_{\lambda-\mu}$ hence $m=\lambda-\mu+1$ and we are done □

Example. For $\mathfrak{g} = \mathfrak{sl}_2$

$$A_s(\lambda) \cdot m_n(k) = (-1)^k \, q^{n-2k} \prod_{j=1}^{k} \frac{[\lambda+j+1]}{[\lambda-n+k+j]} \; m_n(n-k)$$

$$\lim_{q^\lambda \to \infty} A_s(\lambda)\Big|_{L_n} = (-1)^n \, \mathbb{S}\Big|_{L_n} \qquad \lim_{q^\lambda \to 0} A_s(\lambda) = \mathbb{S}^{-1} q^{h}$$

Properties of $A_w(\lambda)$ (follow from definitions and quantum Verma id.)

(1) $A_w(\lambda) : V[\mu] \to V[w\mu]$ is invertible, rational function of q^λ

(2) If $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression then

$$A_w(\lambda) = A_{s_{i_1}}\big((s_{i_2} \cdots s_{i_\ell}) \cdot \lambda\big) \cdots A_{s_{i_\ell}}(\lambda)$$

and $A_{s_i}(\lambda)\Big|_V = A_s(\lambda(h_i))\Big|_V$ restricted to $U_{q_i}\mathfrak{sl}_2$.

Cor. $\{\mathbb{S}_i\}_{i \in I}$ satisfy braid relations.

(7.6) Prop. $$A_{w;\, V_1 \otimes V_2}(\lambda)\, J_{V_1,V_2}(\lambda) = J_{V_1,V_2}(w \cdot \lambda)\left(A_{w,V_1}(\lambda - h^{2}) \otimes A_{w,V_2}(\lambda)\right)$$

Proof follows from the commutative diagram $\forall\, v_i \in V_i[\mu_i]$

$$\begin{tikzcd}
M_\lambda \arrow[r] & M_{\lambda-\mu_2} \otimes V_2 \arrow[r] & M_{\lambda-\mu_1-\mu_2} \otimes V_1 \otimes V_2 \\
M_{w\cdot\lambda} \arrow[r] \arrow[u, hook] & M_{w\cdot(\lambda-\mu_2)} \otimes V_2 \arrow[r] \arrow[u, hook] & M_{w\cdot(\lambda-\mu_1-\mu_2)} \otimes V_1 \otimes V_2 \arrow[u, hook]
\end{tikzcd}$$

□

(7.7) Proof of ABRR equation.

Let $v_i \in V_i[\mu_i]$. Define $X(\lambda)$ by the following ($u = S(b_i)a_i$ is the Drinfeld element)

$$\begin{tikzcd}
M_\lambda \arrow[r, "\varphi_\lambda^{v_2}"] \arrow[drr, dashed] & M_{\lambda-\mu_2}\otimes V_2 \arrow[r, "u\otimes 1"] & M_{\lambda-\mu_2}\otimes V_2 \arrow[d, "\varphi^{v_1}_{\lambda-\mu_2}\otimes \mathrm{id}_{V_2}"] \\
 & & M_{\lambda-\mu_1-\mu_2}\otimes V_1\otimes V_2
\end{tikzcd}$$

$$\mathbb{1}_\lambda \longmapsto \mathbb{1}_{\lambda-\mu_1-\mu_2}\otimes X(\lambda) + \dots$$

Since $C^q = q^{2\rho}u^{-1}$ acts by $q^{(\lambda-\mu_2+2\rho,\,\lambda-\mu_2)}$ on $M_{\lambda-\mu_2}$ we get

$$X(\lambda) = q^{(\lambda,2\rho)-(\lambda-\mu_2+2\rho,\,\lambda-\mu_2)}\,(1\otimes q^{-2\rho})\,J_{V_1V_2}(\lambda)(v_1\otimes v_2)$$

Now we compute $X(\lambda)$ from the definition of $u = S(b_i)a_i$

$$(\varphi^{v_1}_{\lambda-\mu_2}\otimes 1)(S(b_i)a_i\otimes 1)\,\varphi_\lambda^{v_2}(1_\lambda)$$

$$= (\Delta(S(b_i))\otimes 1)(\varphi^{v_1}_{\lambda-\mu_2}\otimes 1)(a_i\otimes 1)\,\varphi^{v_2}_\lambda(1_\lambda) \qquad \text{(using Cabling id.)}$$

$$= (S(b_i)\otimes S(b_j)\otimes 1)(\varphi^{v_1}_{\lambda-\mu_2}\otimes 1)(a_ia_j\otimes 1)\,\varphi^{v_2}_\lambda(1_\lambda)$$

$S(b_i)\in \mathcal{U}^{\leq 0}$ and we are interested in coefficient of $1_{\lambda-\mu_1-\mu_2}$, $X(\lambda) =$

$$= (1\otimes S(b_j)\otimes 1)(q^{-\sum x_a^2}\otimes 1\otimes 1)(q^{-\sum x_a\otimes x_a}\otimes 1)\left[\varphi^{v_1}_{\lambda-\mu_2}\otimes 1\ (a_j\otimes 1)\,\varphi^{v_2}_\lambda(1_\lambda)\right]$$

$$= q^{-\|\lambda-\mu_1-\mu_2\|^2}\left(1\otimes S(b_j)\,q^{-\lambda+\mu_1+\mu_2}\otimes 1\right)\left[\varphi^{v_1}_{\lambda-\mu_2}\otimes 1\ (a_j\otimes 1)\,\varphi_\lambda^{v_2}(1_\lambda)\right]$$

$$= q^{-\|\lambda-\mu_1-\mu_2\|^2}(1\otimes S(b_k)\otimes a_k)^{-1}\left(1\otimes S(b_jb_k)\,q^{-\lambda+\mu_1+\mu_2}\otimes 1\right)\left[\varphi^{v_1}_{\lambda-\mu_2}\otimes 1\ (a_j\otimes a_k)\,\varphi^{v_2}_\lambda(1_\lambda)\right]$$

- $a_j\otimes a_k\otimes b_jb_k = R_{13}R_{23} = \Delta\otimes 1(R)$
- $(S(b_k)\otimes a_k)^{-1} = q^{2\rho}\otimes 1\ R_{21}\ q^{-2\rho}\otimes 1$

$$q^{-\|\lambda-\mu_1-\mu_2\|^2}\left(1\otimes\left(q^{2\rho}\otimes 1\ R_{21}\ q^{-2\rho}\otimes 1\right)\right)\left(1\otimes q^{-2\lambda+\mu_1+\mu_2}\otimes 1\right)\left(\varphi^{v_1}_{\lambda-\mu_2}\otimes 1\ \ \varphi^{v_2}_{\lambda}(1_\lambda)\right)$$

$$\Rightarrow\ X(\lambda) = q^{-\|\lambda-\mu_1-\mu_2\|^2}\left(q^{2\rho}\otimes 1\right)R_{21}\left(q^{-2\rho}\otimes 1\right)\ \cancel{\otimes}\ q^{-2\lambda+\mu_1+\mu_2}\otimes 1\ \ J_{V_1V_2}(\lambda)(v_1\otimes v_2)$$

Comparing the two expressions and weight 0 property of R & J we get

$$J_{V_1V_2}(\lambda)\left(1\otimes q^{2\lambda+2\rho-\sum x_a^2}\right) = R_{21}\ q^{-\sum x_a\otimes x_a}\left(1\otimes q^{2\lambda+2\rho-\sum x_a^2}\right) J_{V_1V_2}(\lambda)$$

Cor
$$\lim_{q^\lambda\to\infty} J(\lambda) = 1 \qquad \lim_{q^\lambda\to 0} J(\lambda) = R_{21}\, q^{-\sum x_a\otimes x_a}$$

Proof.
$$J(\lambda) = 1\otimes q^{-2\theta(\lambda)}\ \underbrace{(R^0_{21})^{-1}\, J(\lambda)}_{\in\ \mathcal{U}^-\otimes\,\mathcal{U}^+}\ 1\otimes q^{2\theta(\lambda)}$$

$$\Rightarrow \lim_{q^\lambda\to\infty} J(\lambda) = 1$$

Similarly
$$J(\lambda) = R^0_{21}\ 1\otimes q^{2\theta(\lambda)}\ \underbrace{J(\lambda)}_{\in\ \mathcal{U}^-\otimes\,\mathcal{U}^+}\ 1\otimes q^{-2\theta(\lambda)}$$

$$\Rightarrow \lim_{q^\lambda\to 0} J(\lambda) = R^0_{21}$$

□