Lecture 8 Quantum affine algebra

(8.0) Let $\mathfrak{g}$ be finite-dimensional simple Lie algebra. We return back to our standard notations.

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra. $A = (a_{ij})_{i,j \in I}$ Cartan matrix $D = (d_i)_{i \in I}$ positive (coprime) integers s.t. $d_i a_{ij} = d_j a_{ji}$. And so on.

$\hat{\mathfrak{g}}$ = untwisted affine Lie algebra

Kac-Moody presentation. $\hat{\mathfrak{g}}$ = KM algebra assoc. to $\hat{A} = (a_{ij})_{i,j \in \hat{I}}$

$$\hat{I} = I \cup \{0\} \qquad a_{00} = 2 \qquad a_{0i} = -\alpha_i(\theta^\vee) \qquad a_{i0} = -\theta(h_i)$$

Loop presentation : $\hat{\mathfrak{g}} = (\mathfrak{g}[z, z^{-1}] \oplus \mathbb{C}c) \rtimes \mathbb{C}d$

(8.1) $U_q(\hat{\mathfrak{g}})$ is then defined as before since $\hat{\mathfrak{g}}$ is a KM algebra

Generators : $h \in \tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, e_i, f_i $(i \in I \cup \{0\})$

Relations : $[h, h'] = 0 \quad \forall\, h, h' \in \tilde{\mathfrak{h}}$ $\qquad [h, e_i] = \alpha_i(h) e_i \qquad [h, f_i] = -\alpha_i(h) f_i$ $(\forall\, i \in \tilde{I})$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \qquad (\forall\, i,j \in \tilde{I})$$

Serre relations : $i \neq j \in \hat{I}$ $\qquad r = 1 - a_{ij}$

$$\sum_{s=0}^{r} (-1)^s e_i^{(r-s)} e_j e_i^{(s)} = 0 = \sum_{s=0}^{r} (-1)^s f_i^{(r-s)} f_j f_i^{(s)}$$

Recall that We have an action of B^a (affine braid group) on $U_q(\hat{\mathfrak{g}})$. extended affine braid group B^e admits the following presentations

I. Generators $\left\{ U_\pi, T_i \;:\; \begin{matrix} \pi \in \Pi \\ i \in \hat{I} \end{matrix} \right\}$ $\qquad$ Relations : braid relation for T_i's

$$U_\pi U_{\pi'} = U_{\pi+\pi'}$$
$$U_\pi T_i U_\pi^{-1} = T_{\pi(i)}$$

II Generators $\left\{ Y_{\lambda^\vee}, T_i \;;\; \begin{matrix} \lambda^\vee \in P^\vee \\ i \in I \end{matrix} \right\}$

Relations : braid relations for T_i's

$$Y_{\lambda^\vee} Y_{\mu^\vee} = Y_{\lambda^\vee + \mu^\vee}$$

if $\alpha_i(\lambda^\vee) = 0$ then $T_i Y_{\lambda^\vee} = Y_{\lambda^\vee} T_i$

if $\alpha_i(\lambda^\vee) = 1$ then $T_i^{-1} Y_{\lambda^\vee} T_i^{-1} = Y_{s_i(\lambda^\vee)}$

Let B^e act on $\mathcal{U}_q\hat{\mathfrak{g}}$ by : $T_i = \mathrm{Ad}(\mathcal{S}_i)$ as before $(i \in \hat{I})$

$$U_\pi \cdot e_i = e_{\pi(i)} \qquad U_\pi f_i = f_{\pi(i)} \qquad U_\pi(h_i) = h_{\pi(i)}$$

Note : U_π extends to $\tilde{\mathfrak{h}}$ by uniqueness of realization of Cartan matrices.

Define
$$F_{i,k} := \Upsilon_{\omega_i^\vee}^{k} f_i \qquad \forall\ i \in I,\ k \in \mathbb{Z}$$
$$E_{i,k} := \Upsilon_{\omega_i^\vee}^{-k} e_i$$

For $r > 0$ set
$$\psi_{i,r} = (q_i - q_i^{-1})\, q^{rc/2}\, [E_{i,0}, F_{i,r}]$$
$$\phi_{i,-r} = -(q_i - q_i^{-1})\, q^{-rc/2}\, [E_{i,-r}, F_{i,0}]$$

$$\psi_{i,0} := k_i = q_i^{h_i} \qquad \phi_{i,0} = k_i^{-1} \qquad (\forall\ i \in I)$$

Define $h_{i,k}$ $(k \in \mathbb{Z} \setminus \{0\})$ by the following equations.

$$\sum_{k \geq 0} \psi_{i,k} z^{-k} = k_i \exp\Big((q_i - q_i^{-1}) \sum_{r \geq 1} h_{i,r} z^{-r}\Big)$$
$$\sum_{k \geq 0} \phi_{i,-k} z^{k} = k_i^{-1} \exp\Big(-(q_i - q_i^{-1}) \sum_{r \geq 1} h_{i,-r} z^{r}\Big)$$

Theorem. $\mathcal{U}_q\hat{\mathfrak{g}}$ is generated by $\{H_{i,k}, E_{i,k}, F_{i,k}\}_{i \in I,\, k \in \mathbb{Z}}$ and c, d subject to the following relations

(0) c is central $\quad [d, \Upsilon_k] = k\,\Upsilon_k \quad$ (for $\Upsilon = H_i, E_i$ or F_i)

$$[H_{i,0}, H_{j,k}] = 0 \qquad [H_{i,0}, E_{j,k}] = a_{ij} E_{j,k} \qquad [H_{i,0}, F_{j,k}] = -a_{ij} F_{j,k}$$

(1)
$$[H_{i,k}, H_{j,l}] = \delta_{k,-l} \frac{[k a_{ij}]_i}{k} \frac{q^{kc} - q^{-kc}}{q_j - q_j^{-1}}$$

(2)
$$[H_{i,k}, E_{j,l}] = \frac{[k a_{ij}]_i}{k} q^{-\frac{|k|}{2}c} E_{j,k+l}$$
$$[H_{i,k}, F_{j,l}] = -\frac{[k a_{ij}]_i}{k} q^{+\frac{|k|}{2}c} F_{j,k+l}$$

(3) $$E_{i,k+1} E_{j,\ell} - q_i^{a_{ij}} E_{j,\ell} E_{i,k+1} = q_i^{a_{ij}} E_{i,k} E_{j,\ell+1} - E_{j,\ell+1} E_{i,k}$$

$$F_{i,k+1} F_{j,\ell} - q_i^{-a_{ij}} F_{j,\ell} F_{i,k+1} = q_i^{-a_{ij}} F_{i,k} F_{j,\ell+1} - F_{j,\ell+1} F_{i,k}$$

(4) $$[E_{i,k}, F_{j,\ell}] = \delta_{ij} \frac{q^{\frac{k-\ell}{2}c} \psi_{i,k+\ell} - q^{-\frac{k-\ell}{2}c} \phi_{i,k+\ell}}{q_i - q_i^{-1}}$$

(5) Let $i \neq j \in I$, $n = 1 - a_{ij}$, $k_1 \dots k_n \in \mathbb{Z}$, $\ell \in \mathbb{Z}$.

$$\sum_{\substack{r=0\dots n \\ \pi \in S_n}} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix}_i E_{i,k_{\pi(1)}} \dots E_{i,k_{\pi(r)}} E_{j,\ell} E_{i,k_{\pi(r+1)}} \dots E_{i,k_{\pi(n)}} = 0$$

(same with F's instead of E's)

(8.2) We will be studying finite-dimensional representations V of $\mathcal{U}_q \hat{\mathfrak{g}}$,

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu] \quad , \dim V < \infty .$$

Remark: $\mathcal{U}_q \hat{\mathfrak{g}}$ has no non-trivial f.d. repns. Since if V is such and if $v \in V$ $(v \neq 0)$ is an eigenvector for d $(dv = \lambda v)$ then $d E_{i,k} v = (\lambda + k) E_{i,k} v \Rightarrow \{E_{i,k} v\}_{k \in \mathbb{Z}}$ are l.i. contradicting finite-dimensionality of V.

So we only consider algebra $\mathcal{U}'$ generated by $H_{i,k}$, $E_{i,k}$, $F_{i,k}$ and c.

Lemma. If V is a f.d. repn of $\mathcal{U}'$, then c acts by 0 on V.

Proof Let $V^0 = \{ v \in V \mid E_{i,k} v = 0 \ \forall i \in I, k \in \mathbb{Z} \}$

Then V is generated by V^0. We claim $cv = 0 \ \forall v \in V^0$

As $U_q\mathfrak{g}$-module V has weight space dec and similarly

$$V^0 = \bigoplus_{\mu\in\mathfrak{h}^*} V^0[\mu] \quad \text{and} \quad c \text{ preserves this decomposition. Let } v\in V^0[\mu]$$

be an eigenvector for c $\quad cv = sv \quad (s\in\mathbb{C}). \quad m_i = \mu(h_i) \geq 0.$

Then $\forall\, r\in\mathbb{Z}$, v is a h.w. vector for $U_{q_i}sl_2 = \langle q^{rc}K_i, E_{i,r}, F_{i,-r}\rangle$

$\Rightarrow \quad rs + m_i \geq 0 \quad \forall\, r\in\mathbb{Z},\ i\in I$. Hence $s=0$. $\square$

Define $\quad U_q(L\mathfrak{g}) := \mathcal{U}'/\langle c\rangle$

Generators : $\{H_{i,k},\ E_{i,k},\ F_{i,k} \ : \ i\in I,\ k\in\mathbb{Z}\}$

Relations : (0) $\quad [H_{i,k}\ H_{j,l}] = 0 \quad [H_{i,0}\ E_{j,k}] = a_{ij}E_{j,k} \quad [H_{i,0}\ F_{j,k}] = -a_{ij}F_{j,k}$

(1) $$[H_{i,r}\ E_{j,k}] = \frac{[ra_{ij}]_i}{r} E_{j,k+r} \qquad [H_{i,r}\ F_{j,k}] = -\frac{[ra_{ij}]_i}{r} F_{j,k+r}$$

(2) $$E_{i,k+1}E_{j,l} - q_i^{a_{ij}} E_{j,l}E_{i,k+1} = q_i^{a_{ij}} E_{i,k}E_{j,l+1} - E_{j,l+1}E_{i,k}$$

$$F_{i,k+1}F_{j,l} - q_i^{-a_{ij}} F_{j,l}F_{i,k+1} = q_i^{-a_{ij}} F_{i,k}F_{j,l+1} - F_{j,l+1}F_{i,k}$$

(3) $$[E_{i,k}, F_{j,l}] = \delta_{ij} \frac{\psi_{i,k+l} - \phi_{i,k+l}}{q_i - q_i^{-1}}$$

(4) $\quad i\neq j \quad n = 1-a_{ij}.\ \forall\, k_1\ldots k_n,\ l\in\mathbb{Z}$:

$$\sum_{\substack{r=0\ldots n\\ \pi\in S_n}} (-1)^r \begin{bmatrix} n\\ r\end{bmatrix}_i E_{i,k_{\pi(1)}}\cdots E_{i,k_{\pi(r)}} E_{j,l} E_{i,k_{\pi(r+1)}}\cdots E_{i,k_{\pi(n)}} = 0$$

We will write ψ^+ for ψ and ψ^- for ϕ from now on.

(8.3) Fields

$$\psi_i^{\pm}(z) = \sum_{k\geq 0} \psi^{\pm}_{i,\pm k}\, z^{\mp k} \qquad E_i(z) = \sum_{\ell\in\mathbb{Z}} E_{i,\ell}\, z^{-\ell} \qquad F_i(z) = \sum_{\ell\in\mathbb{Z}} F_{i,\ell}\, z^{-\ell}$$

Prop. (a) $\left[\psi_i^{\varepsilon_1}(z), \psi_j^{\varepsilon_2}(w)\right] = 0 \qquad (\varepsilon_1,\varepsilon_2 \in \{\pm\}) \quad \forall\ i,j\in I$

(b) $$\left[E_i(z), F_j(w)\right] = \delta_{ij}\, \delta(zw^{-1})\, \frac{\psi_i^+(z) - \psi_i^-(z)}{q_i - q_i^{-1}} \qquad \forall\ i,j\in I$$

(c) $$E_i(z)\, E_j(w) = \frac{q_i^{a_{ij}} z - w}{z - q_i^{a_{ij}} w}\, E_j(w)\, E_i(z) \qquad \left[\delta(x) = \sum_{n\in\mathbb{Z}} x^n\right]$$

$$F_i(z)\, F_j(w) = \frac{z - q_i^{a_{ij}} w}{q_i^{a_{ij}} z - w}\, F_j(w)\, F_i(z)$$

(d) $$\psi_i^{\pm}(z)\, E_j(w)\, \psi_i^{\pm}(z)^{-1} = \frac{q_i^{a_{ij}} z - w}{z - q_i^{a_{ij}} w}\, E_j(w)$$

$$\psi_i^{\pm}(z)\, F_j(w)\, \psi_i^{\pm}(z)^{-1} = \frac{z - q_i^{a_{ij}} w}{q_i^{a_{ij}} z - w}\, F_j(w)$$

Proof. (a) is clear. (c) is clearly equivalent to relation (2) of (8.2)

Let us check (b).

$$\left[E_i(z), F_j(w)\right] = \delta_{ij} \sum_{k,\ell\in\mathbb{Z}} \frac{\psi^+_{i,k+\ell} - \psi^-_{i,k+\ell}}{q_i - q_i^{-1}}\, z^{-k} w^{-\ell}$$

$$= \frac{\delta_{ij}}{q_i - q_i^{-1}} \sum_{p\, \cancel{\in\mathbb{Z}}\, \geq 0} \psi^+_{i,p}\, z^{-p} \left(\sum_{n\in\mathbb{Z}} z^n w^{-n}\right) - \psi^-_{i,-p}\, z^{p} \left(\sum_{n\in\mathbb{Z}} z^n w^{-n}\right)$$

$$= \frac{\delta_{ij}}{q_i - q_i^{-1}}\, \delta(zw^{-1}) \left(\psi_i^+(z) - \psi_i^-(z)\right)$$

(d) (only for + case. − case is similar; also only for E's)

$$\psi_i(z)\, E_j(w)\, \psi_i(z)^{-1} = \frac{q_i^{a_{ij}} z - w}{z - q_i^{a_{ij}} w} E_j(w)$$

$$\equiv \quad \mathrm{Ad}\left(\psi_{i,0}^{-1}\psi_i(z)\right).\ E_j(w) = \frac{z - q_i^{-a_{ij}} w}{z - q_i^{a_{ij}} w} E_j(w) \qquad \left(\text{since } \psi_{i,0}^{+} = K_i \quad K_i E_{j,l} K_i^{-1} = q_i^{a_{ij}} E_{j,l}\right)$$

$$\equiv \quad \mathrm{ad}\left((q_i - q_i^{-1})\sum_{n\geq 1} H_{i,n} z^{-n}\right) E_j(w) = \log\left(\frac{1 - q_i^{-a_{ij}} w z^{-1}}{1 - q_i^{a_{ij}} w z^{-1}}\right) E_j(w)$$

$$= \left[\sum_{n\geq 1} \frac{q_i^{n a_{ij}} - q_i^{-n a_{ij}}}{n} w^n z^{-n}\right] E_j(w)$$

$$\equiv \quad \forall\, n \geq 1 \qquad (q_i - q_i^{-1})\left[H_{i,n}\ E_{j,l}\right] = \frac{q_i^{n a_{ij}} - q_i^{-n a_{ij}}}{n} E_{j,l+n} \qquad \square$$

(8.4) Half fields

$$E_i^+(z) = \sum_{k\geq 0} E_{i,k} z^{-k} \qquad E_i^-(z) = -\sum_{k\geq 1} E_{i,-k} z^{k}$$

$$F_i^+(z) = \sum_{k\geq 0} F_{i,k} z^{-k} \qquad F_i^-(z) = -\sum_{k\geq 1} F_{i,-k} z^{k}$$

Prop. Let (π, V) be a f.d. repn of $\mathcal{U}_q(L\mathfrak{g})$. Then $\exists$ rat'l functions, End(V)-valued, $\{\psi_i(z),\ e_i(z),\ f_i(z)\}_{i\in I}$ s.t

$$\pi\left(\psi_i^{\pm}(z)\right) = \left[\psi_i(z)\right]^{\pm} \qquad \pi\left(E_i^{\pm}(z)\right) = \left[e_i(z)\right]^{\pm} \qquad \pi\left(F_i^{\pm}(z)\right) = \left[f_i(z)\right]^{\pm}$$

when $[f(z)]^{\pm}$ denotes expansion near $z = \infty$ and 0 respectively.

Proof.
$$\left[H_{i,1},\ E_i^+(z)\right] = [2]_i \left(\sum_{k\geq 0} E_{i,k+1} z^{-k-1}\right) z$$

$$= [2]_i\ z\ \left(E_i^+(z) - E_{i,0}\right)$$

$$\Rightarrow \left(z - \frac{\mathrm{ad}\, H_{i,1}}{[2]_i} \right) E_i^+(z) = z\, E_{i,0}$$

$\Rightarrow E_i^+(z) = \left(z - \frac{\mathrm{ad}\, H_{i,1}}{[2]_i} \right)^{-1} z\, E_{i,0}$ is rat'l fn regular at ∞.

Similarly $E_i^-(z) = -\left(1 - \frac{\mathrm{ad}\, H_{i,-1}}{[2]_i} z \right)^{-1} z\, E_{i,-1}$ is rat'l fn. regular at 0.

Moreover $E_i^-(z)$ also satisfies $\left(z - \frac{\mathrm{ad}\, H_{i,1}}{[2]_i} \right) E_i^-(z) = z\, E_{i,0}$. Hence $E_i^{\pm}(z)$ are expansions of the same function. Similarly for $F_i^{\pm}(z)$.

Finally we have $\psi_i^+(z) = \left[E_i^+(z), F_{i,0} \right] (q_i - q_i^{-1}) + \psi_{i,0}^-$ and

$$\psi_i^-(z) = \left[E_i^-(z), F_{i,0} \right] (q_i - q_i^{-1}) + \psi_{i,0}^- .$$

□

(8.5) Relations in half fields:

(1) $[\psi_i(z), \psi_j(w)] = 0 \quad \forall\, i,j \in I$

(2) $$\psi_i(z)\, e_j(w)\, \psi_i(z)^{-1} = \frac{q_i^{a_{ij}} z - w}{z - q_i^{a_{ij}} w}\, e_j(w) - \frac{\left(q_i^{a_{ij}} - q_i^{-a_{ij}}\right) q_i^{a_{ij}} w}{z - q_i^{a_{ij}} w}\, e_j(q_i^{-a_{ij}} z)$$

(similarly for f_j with $q_i^{a_{ij}}$ replaced by $q_i^{-a_{ij}}$)

(3) $$e_i(z)\, e_j(w) = \frac{q_i^{a_{ij}} z - w}{z - q_i^{a_{ij}} w}\, e_j(w)\, e_i(z) + \frac{\left[z\left(E_{i,0}\, e_j(w) - q_i^{a_{ij}} e_j(w) E_{i,0}\right) + w\left(E_{j,0}\, e_i(z) - q_i^{a_{ij}} e_i(z) E_{j,0}\right) \right]}{z - q_i^{a_{ij}} w}$$

(4) $$\left[e_i(z), f_j(w) \right] = \frac{\delta_{ij}}{(q_i - q_i^{-1})(z-w)} \left[z\, \psi_i(w) - w\, \psi_i(z) - (z-w)\, \psi_{i,0}^- \right]$$

(8.6) Shift automorphism : $\forall \zeta \in \mathbb{C}^\times$ we have the following automorphism of $U_q(L\mathfrak{g})$ (as a Hopf algebra)

$$\tau_\zeta(Y_k) = \zeta^k Y_k \quad (\forall k \in \mathbb{Z}) \quad \text{where } Y = H_i, E_i \text{ or } F_i.$$

In terms of generating series (or fields)

$$\tau_\zeta(Y(z)) = Y(\zeta^{-1} z).$$ This is a one-parameter group of automorphisms. For $V \in \mathrm{Rep}_{fd}(U_q(L\mathfrak{g}))$ we denote by $V(\zeta)$ the pull-back representation $\tau_\zeta^* V$.

(8.7) Classification of finite dim'l irreducible representations

Let V be an irr. f.d. repn of $U_q(L\mathfrak{g})$. Let $V^0 = \{v \in V \mid E_{i,k} v = 0 \ \forall i \in I, k \in \mathbb{Z}\}$

$\{\psi^\pm_{i,\pm k}\}$ preserve this. Let $v \in V^0$ be a joint eigenvector of $\{\psi^\pm_{i,\pm k}\}$.

Then there are rat'l fns $\psi_i(z)$ regular at 0 and ∞ s.t.

$$\psi_i^\pm(z)\, v = \psi_i(z)\, v \quad \forall i \in I$$

Theorem. $\exists!$ monic polynomials $P_i(w) \in \mathbb{C}[w]$ $\quad P_i(0) \neq 0$ s.t

$$\psi_i(z) = \frac{P_i(q_i^2 z)}{P_i(z)} \cdot q_i^{-\deg P_i} \quad \forall i \in I$$

Conversely every $|I|$-tuple of such polynomials gives rise to a unique f.d. irr. repn.

$$\begin{array}{c}\text{Irr. f.d. repns of } U_q(L\mathfrak{g}) \\ \text{(up to iso.)}\end{array} \longleftrightarrow \left\{ \left\{ \begin{array}{c} P_i(w) \in \mathbb{C}[w] : \\ P_i \text{ is monic} \\ P_i(0) \neq 0 \end{array} \right\}_{i \in I} \right\}$$

Proof will follow from future constructions.

(8.8) Example of $\mathfrak{g} = \mathfrak{sl}_2$. Generators $\{H_k, E_k, F_k\}_{k \in \mathbb{Z}}$

(i) Coproduct.

$$\Delta(H_1) = H_1 \otimes 1 + 1 \otimes H_1 - (q^2 - q^{-2}) E_0 \otimes F_1$$

$$\Delta(H_{-1}) = H_{-1} \otimes 1 + 1 \otimes H_{-1} + (q^2 - q^{-2}) E_{-1} \otimes F_0$$

$$\Delta(E_0) = E_0 \otimes K + 1 \otimes E_0 \qquad \Delta(F_0) = F_0 \otimes 1 + K^{-1} \otimes F_0$$

$$\Delta(H_0) = H_0 \otimes 1 + 1 \otimes H_0$$

(ii) Evaluation : $ev_\zeta : \mathcal{U}_q(L\mathfrak{sl}_2) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2)$

$$E_0 \; F_0 \; H_0 \longrightarrow E_0, F_0, H_0 = e, f, h$$

$$E_{-1} \longrightarrow q \zeta^{-1} k^{-1} e$$

$$F_1 \longrightarrow q^{-1} \zeta f k$$

$\mathcal{U}_q(L\mathfrak{sl}_2) \curvearrowright \mathbb{C}^2_\zeta$ by

$$e(z)|\downarrow\rangle = \frac{z}{z-\zeta}|\uparrow\rangle \qquad \psi(z)|\uparrow\rangle = \frac{qz - q^{-1}\zeta}{z-\zeta}|\uparrow\rangle$$

$$f(z)|\uparrow\rangle = \frac{z}{z-\zeta}|\downarrow\rangle \qquad \psi(z)|\downarrow\rangle = \frac{q^{-1}z - q^{+1}\zeta}{z-\zeta}|\downarrow\rangle$$

Drinfeld poly $= z - \zeta$.