(9.0) Recall: for a finite-dimensional simple Lie algebra $\mathfrak{g}$, we defined the quantum affine algebra $\mathcal{U}_q\hat{\mathfrak{g}}$ as the quantum group associated to the Kac-Moody algebra $\hat{\mathfrak{g}}$. We introduced the Drinfeld's new presentation of $\mathcal{U}_q\hat{\mathfrak{g}}$ and $\mathcal{U}_q(L\mathfrak{g})$ (quantum loop algebra) as a sub-quotient of $\mathcal{U}_q\hat{\mathfrak{g}}$:

$\mathcal{U}'$ = subalg. gen. by all the other generators except $d$

$$\mathcal{U}_q(L\mathfrak{g}) = \mathcal{U}'/\langle c\rangle .$$

Summary

Kac-Moody presentation $\{e_i, f_i, h_i\}_{i\in\hat{I}}$ $\overset{\text{Beck's iso.}}{\rightsquigarrow}$ Loop presentation $\{E_{i,k}\ H_{i,k}\ F_{i,k}\}_{i\in I, k\in\mathbb{Z}}$

$$\Big( c = h_0 + \sum_{i\in I} m_i h_i = 0 \Big)$$

Isomorphism between the two presentations uses the action of extended affine braid group.

We also introduced the generating series (or fields)

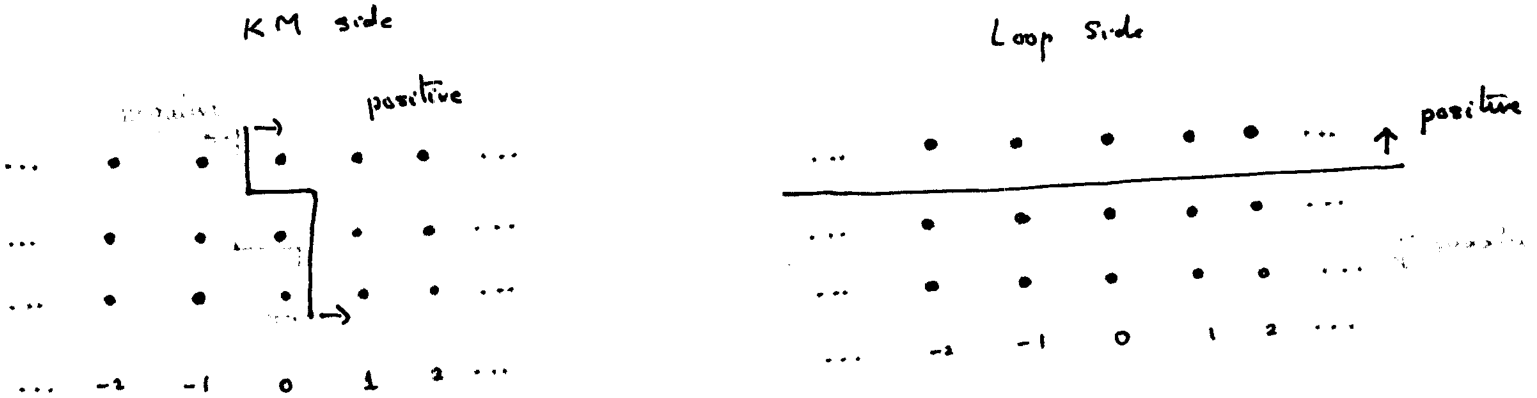
$$\psi_i^{\pm}(z) = \sum_{k\geq 0} \psi^{\pm}_{i,\pm k}\, z^{\mp k} = K_i^{\pm 1} \exp\Big(\pm (q_i - q_i^{-1}) \sum_{n\geq 1} H_{i,\pm n}\, z^{\mp n}\Big)$$

$$E_i(z) = \sum_{n\in\mathbb{Z}} E_n z^{-n} = E_i^+(z) - E_i^-(z) \quad \text{where} \quad E_i^+(z) = \sum_{n\geq 0} E_n z^{-n} \quad (\text{similarly } F_i)$$

$$E_i^-(z) = -\sum_{n\geq 1} E_{-n} z^{n}$$

Rationality: on a f.d. repn, $\psi_i^{\pm}(z)$ (resp. $E_i^{\pm}(z)$, $F_i^{\pm}(z)$) are expansions at $\infty$ and $0$ of a rational function $\psi_i(z)$ (resp $e_i(z)$, $f_i(z)$).

(9.1) Picture




Shift automorphism. $\forall\, \zeta \in \mathbb{C}^\times$, there is an automorphism $\tau_\zeta$ of $\mathcal{U}_q(L\mathfrak{g})$ (as a Hopf algebra) s.t. $\tau_{\zeta_1}\tau_{\zeta_2} = \tau_{\zeta_1\zeta_2}$ and $\tau_0 = \mathrm{id}$.

Kac-Moody side: $\tau_\zeta(e_i) = e_i \quad \tau_\zeta(f_i) = f_i \quad \forall\, i \in I \qquad \tau_\zeta(h) = h \quad \forall\, h \in \mathfrak{h}$.

$$\tau_\zeta(e_0) = \zeta e_0 \qquad \tau_\zeta(f_0) = \zeta^{-1} f_0$$

Loop side: $\tau_\zeta(Y_k) = \zeta^k Y_k \qquad (\forall\ k \in \mathbb{Z},\ Y = H_i, E_i \text{ or } F_i \ (i \in I))$.

in terms of generating series $\quad \tau_\zeta(Y^\pm(z)) = Y^\pm(\zeta^{-1} z)$

(9.2) Explicit example of $sl_2$.

Kac-Moody presentation

$e_0,\ h_0,\ f_0$

$e_1,\ h_1,\ f_1$

$(c = h_0 + h_1 = 0)$

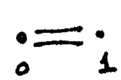
Cartan matrix $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$$B^e = \langle Y, T \mid T Y^{-1} T = Y \rangle$$

$$= \left\langle \pi, T_0, T_1 \;\middle|\; \begin{matrix} \pi T_0 \pi^{-1} = T_1 \\ \pi^2 = 1 \end{matrix} \right\rangle$$

$$\left.\begin{matrix} Y = \pi T_1 \\ T = T_1 \end{matrix}\right\} \text{ iso.}$$

Dynkin diagram




$B^e$ action on $\mathcal{U}_q'\widehat{sl_2}$ :

$T_1(e_1) = -f_1 k_1 \qquad T_1(f_1) = -k_1^{-1} e_1 \qquad T_1(h_1) = -h_1$

$\pi(e_1) = e_0 \qquad \pi(f_1) = f_0 \qquad \pi(h_1) = h_0 \ (= -h_1)$

$$T_1(e_0) = e_1^{(2)} e_0 - q^{-1} e_1 e_0 e_1 + q^{-2} e_0 e_1^{(2)}$$

$$T_1(f_0) = f_0 f_1^{(2)} - q\, f_1 f_0 f_1 + q^2 f_1^{(2)} f_0$$

Isomorphism. KM $\longrightarrow$ Loop is computed as

$$E_0 = e_1 \qquad F_0 = f_1 \qquad H_0 = h_1 \qquad \leftarrow \text{ copy of original } U_q sl_2.$$

$$E_{-1} = Y \cdot e_1 = \pi(T_1 e_1) = -f_0 k_1^{-1}$$

$$F_1 = Y \cdot f_1 = \pi(T_1 f_1) = -k_1 e_0$$

Transfering coproduct: $\Delta(E_{-1}) = -(f_0 \otimes 1 + k_0^{-1} \otimes f_0)(k_1^{-1} \otimes k_1^{-1})$

$$= E_{-1} \otimes k_1^{-1} + 1 \otimes E_{-1}$$

$$\Delta(F_1) = -k_1 \otimes k_1 (e_0 \otimes k_0 + 1 \otimes e_0) = F_1 \otimes 1 + k_1 \otimes F_1 .$$

$$\Rightarrow \quad \Delta(H_1) = H_1 \otimes 1 + 1 \otimes H_1 - (q^2 - q^{-2}) E_0 \otimes F_1$$

$$\Delta(H_{-1}) = H_{-1} \otimes 1 + 1 \otimes H_{-1} + (q^2 - q^{-2}) E_{-1} \otimes F_0$$

Evaluation homomorphism: for each $\zeta \in \mathbb{C}^\times$, we have $ev_\zeta : U_q(Lsl_2) \to U_q(sl_2)$

algebra hom given by $ev_\zeta(E_{-1}) = q\zeta^{-1} k^{-1} e$

$$ev_\zeta(F_1) = q^{-1} \zeta f k$$

Pull-back representations: let $n \in \mathbb{N}$ and let $L_n$ be irr. $(n+1)$-dim'l $U_q sl_2$-repn:

$L_n(\zeta) = ev_\zeta^* L_n$ is then a f.d. repn. of $U_q(Lsl_2)$.

Compute how the generating series act.

Step 1. $KH_1 = [E_0, F_1]$. $\quad H_1 m(r) = \dfrac{\zeta}{q - q^{-1}} \left( -q^{n+1} - q^{-n-1} + [2] q^{n-2r} \right) m(r)$

$\Rightarrow$ For an operator $X : \mathbb{C}m(r) \to \mathbb{C}m(r-1)$

$$\frac{\mathrm{ad}\, H_1}{[2]} X = \zeta q^{n-2r+1} X$$

Step 2. $e(z) = \left( z - \dfrac{\mathrm{ad}\, H_1}{[2]} \right)^{-1} z E_0 \Rightarrow e(z) m(r) = \dfrac{[n-r+1]\, z}{z - \zeta q^{n-2r+1}} m(r-1)$

Similarly $f(z) m(r) = \dfrac{[r+1]\, z}{z - \zeta q^{n-2r-1}} m(r+1)$

Final step $\psi(z) = \psi_0^- + [e(z), f_0] \Rightarrow \psi(z) m(r) = q^{n-2r} \dfrac{(z - q^{n+1}\zeta)(z - q^{-n-1}\zeta)}{(z - q^{n-2r-1}\zeta)(z - q^{n-2r+1}\zeta)} m(r)$

Tensor product $\mathbb{C}^2_{\zeta_1} \otimes \mathbb{C}^2_{\zeta_2}$ : basis $|\uparrow\uparrow\rangle$ $|\uparrow\downarrow\rangle$ $|\downarrow\uparrow\rangle$ $|\downarrow\downarrow\rangle$

From previous example $(n=1)$ we have

$$H_1 |\uparrow\rangle_\zeta = \frac{\zeta}{q-q^{-1}} \left(-q^2 - q^{-2} + (q+q^{-1})q\right) |\uparrow\rangle_\zeta = q^{-1}\zeta |\uparrow\rangle_\zeta$$

$$H_1 |\downarrow\rangle_\zeta = \frac{\zeta}{q-q^{-1}} \left(-q^2 - q^{-2} + (q+q^{-1})q^{-1}\right) |\downarrow\rangle_\zeta = -q\zeta |\downarrow\rangle_\zeta$$

$$|\uparrow\rangle \underset{E_0}{\overset{F_0}{\rightleftarrows}} |\downarrow\rangle \quad \text{and} \quad E_{-1} |\downarrow\rangle_\zeta = \zeta^{-1} |\uparrow\rangle_\zeta \qquad F_1 |\uparrow\rangle_\zeta = \zeta |\downarrow\rangle_\zeta .$$

$$H_1 |\uparrow\downarrow\rangle = \left(q^{-1}\zeta_1 - q\zeta_2\right) |\uparrow\downarrow\rangle$$

$$H_1 |\downarrow\uparrow\rangle = \left(-q\zeta_1 + q^{-1}\zeta_2\right) |\downarrow\uparrow\rangle - (q^2 - q^{-2})\zeta_2 |\uparrow\downarrow\rangle$$

$$H_1 |\uparrow\uparrow\rangle = q^{-1}(\zeta_1 + \zeta_2) |\uparrow\uparrow\rangle \qquad H_1 |\downarrow\downarrow\rangle = -q(\zeta_1 + \zeta_2) |\downarrow\downarrow\rangle$$

For an operator $X : \mathbb{C}|\downarrow\downarrow\rangle \to \mathbb{C}|\uparrow\downarrow\rangle \oplus \mathbb{C}|\downarrow\uparrow\rangle$ (say $X|\downarrow\downarrow\rangle = x_1|\uparrow\downarrow\rangle + x_2|\downarrow\uparrow\rangle$)

$$\frac{\mathrm{ad}\, H_1}{[2]} . X |\downarrow\downarrow\rangle = \left(\zeta_1 x_1 - (q-q^{-1})\zeta_2 x_2\right) |\uparrow\downarrow\rangle + \zeta_2 x_2 |\downarrow\uparrow\rangle$$

$$\Rightarrow \quad e(z) |\downarrow\downarrow\rangle = \frac{z\left(q^{-1}z - q\zeta_2\right)}{(z-\zeta_1)(z-\zeta_2)} |\uparrow\downarrow\rangle + \frac{z}{z-\zeta_2} |\downarrow\uparrow\rangle$$

Similarly for an operator $X : \mathbb{C}|\uparrow\downarrow\rangle \oplus \mathbb{C}|\downarrow\uparrow\rangle \to \mathbb{C}|\uparrow\uparrow\rangle$ (say $X|\uparrow\downarrow\rangle = x_1|\uparrow\uparrow\rangle$, $X|\downarrow\uparrow\rangle = x_2|\uparrow\uparrow\rangle$)

$$\frac{\mathrm{ad}\, H_1}{[2]} . X : \quad |\uparrow\downarrow\rangle = \zeta_2 x_1 |\uparrow\uparrow\rangle$$

$$|\downarrow\uparrow\rangle = \left(\zeta_1 x_2 + (q-q^{-1})\zeta_2 x_1\right) |\uparrow\uparrow\rangle$$

$$\Rightarrow \quad e(z) |\uparrow\downarrow\rangle = \frac{z}{z-\zeta_2} |\uparrow\uparrow\rangle \qquad e(z) |\downarrow\uparrow\rangle = \frac{z\left(qz - q^{-1}\zeta_2\right)}{(z-\zeta_1)(z-\zeta_2)} |\uparrow\uparrow\rangle$$

(9.3) Classification of finite-dimensional irreducible representations.

Let $\mathcal{V}$ be a f.d. irr. repn of $\mathcal{U}_q(L\mathfrak{g})$. Let $\mathcal{V}^\circ = \{ v \in \mathcal{V} \mid E_{i,k} v = 0 \ \forall \, {i \in I \atop k \in \mathbb{Z}} \}$

Since $\{ \psi^{\pm}_{i, \pm k} \}$ preserve this, we can find a common eigenvector for these

$$0 \neq v \in \mathcal{V}^\circ \qquad \psi^{\pm}_{i, \pm k} v = \lambda^{\pm}_{i, \pm k} v \qquad ( \lambda^{\pm}_{i, \pm k} \in \mathbb{C} )$$

We know that $\psi^{\pm}_i(z) v = (\text{rat'l fn, say } \psi_i(z)) \, v$

<u>Thm.</u> $\exists !$ monic polynomials $P_i(w) \in \mathbb{C}[w]$ ; $P_i(0) \neq 0$ s.t.

$$\psi_i(z) \, v = q_i^{-\deg P_i} \frac{P_i(q_i^2 z)}{P_i(z)} \, v \qquad \forall \, i \in I$$

Conversely every $I$-tuple of monic polynomials with non zero constant term gives rise to a unique f.d. irr. repn. of $\mathcal{U}_q(L\mathfrak{g})$

Summary.

Iso. classes of irr. f.d. repns. of $\mathcal{U}_q(L\mathfrak{g})$ $\longleftrightarrow$ $\mathbb{C}_0[w]^I$ where

$$\mathbb{C}_0[w] = \{ P \in \mathbb{C}[w] : P \text{ is monic} , \ P(0) \neq 0 \}$$

Proof will follow from future constructions.

Example: $L_n(\zeta) \in \mathrm{Rep}_{f.d.}(\mathcal{U}_q(L\mathfrak{sl}_2))$ is irr

$$\psi(z) m_n(0) = q^n \frac{z - q^{-n-1}\zeta}{z - q^{n-1}\zeta} m_n(0)$$

$$= q^n \frac{(z - q^{n-3}\zeta) \cdots (z - q^{-n-1}\zeta)}{(z - q^{n-1}\zeta) \cdots (z - q^{-n+1}\zeta)} = q^{-n} \frac{(q^2 z - q^{n-1}\zeta) \cdots (q^2 z - q^{-n+1}\zeta)}{(z - q^{n-1}\zeta) \cdots (z - q^{-n+1}\zeta)}$$

$$\text{Drinfeld poly} = \prod_{r=0}^{n-1} (z - q^{n-1-2r}\zeta)$$

(9.4) $\mathrm{Rep}_{fd}(\mathcal{U}_q(L\mathfrak{g}))$ is not semisimple.

Example. $\mathfrak{g} = \mathfrak{sl}_2$

(1) Consider the repn. $\mathbb{C}^2_\zeta \otimes \mathbb{C}^2_{q^2\zeta}$. Let $v = -q|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$

$$H_1 v = \left(-q(q^{-1}\zeta - q^3\zeta) - (q^2 - q^{-2})q^2\zeta\right)|\uparrow\downarrow\rangle + (-q\zeta + q\zeta)|\downarrow\uparrow\rangle$$

$$= 0$$

$$e(z)\,v = \left(\frac{-qz}{z - q^2\zeta} + \frac{qz(z-\zeta)}{(z-\zeta)(z-q^2\zeta)}\right)|\uparrow\uparrow\rangle = 0$$

Similarly $f(z)v = 0$ and $\psi(z)v = v$. Thus $\mathbb{C}v$ is (trivial) subrepn of $\mathbb{C}^2_\zeta \otimes \mathbb{C}^2_{q^2\zeta}$. Let $V$ = quotient repn ( $|\downarrow\uparrow\rangle = q|\uparrow\downarrow\rangle$)

We get
$$e(z)|\downarrow\downarrow\rangle = \left[\frac{qz}{z-q^2\zeta} + \frac{z(q^{-1}z - q^3\zeta)}{(z-\zeta)(z-q^2\zeta)}\right]|\uparrow\downarrow\rangle$$

$$= \frac{qz^2 - qz\zeta + q^{-1}z^2 - q^3z\zeta}{(z-\zeta)(z-q^2\zeta)}|\uparrow\downarrow\rangle = \frac{(z-q^2\zeta)[2]z}{(z-\zeta)(z-q^2\zeta)}|\uparrow\downarrow\rangle$$

$$= \frac{[2]\,z}{z-\zeta}|\uparrow\downarrow\rangle$$

and $e(z)|\uparrow\downarrow\rangle = \dfrac{z}{z-q^2\zeta}|\uparrow\uparrow\rangle$.

$\Rightarrow \quad V \simeq L_2(q\zeta)$. We get a non-split short exact sequence

$$0 \to \underset{\substack{\uparrow \\ \text{trivial repn}}}{L_0} \longrightarrow \mathbb{C}^2_\zeta \otimes \mathbb{C}^2_{q^2\zeta} \longrightarrow \underset{\substack{\wr\wr \\ L_2(q\zeta)}}{V} \to 0$$

(2) Similarly consider $\mathbb{C}^2_{q^2\zeta} \otimes \mathbb{C}^2_\zeta$

Check : $\{|\downarrow\downarrow\rangle,\ q^{-1}|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle,\ |\uparrow\uparrow\rangle\}$ Span a subrepn $\simeq L_2(q\zeta)$

Quotient is trivial repn

$$0 \to L_2(q\zeta) \to \mathbb{C}^2_{q^2\zeta} \otimes \mathbb{C}^2_{\zeta} \to L_0 \to 0$$

another non-split short exact sequence

Cor. $\mathbb{C}^2_{\zeta} \otimes \mathbb{C}^2_{q^2\zeta} \not\cong \mathbb{C}^2_{q^2\zeta} \otimes \mathbb{C}^2_{\zeta}$

How to fix this?

(3) See the example of $\mathbb{C}^2_{\zeta_1} \otimes \mathbb{C}^2_{\zeta_2}$. Let us try to construct an iso. $\mathbb{C}^2_{\zeta_1} \otimes \mathbb{C}^2_{\zeta_2} \xrightarrow{\sim} \mathbb{C}^2_{\zeta_2} \otimes \mathbb{C}^2_{\zeta_1}$ which maps highest weight (resp. lowest weight) vector to h.w. (resp. l.w. vector)

$$|\downarrow_{\zeta_1}\downarrow_{\zeta_2}\rangle \longmapsto |\downarrow_{\zeta_2}\downarrow_{\zeta_1}\rangle$$

Apply $E_0$ and $E_1$ to both sides

$$q^{-1}|\uparrow_{\zeta_1}\downarrow_{\zeta_2}\rangle + |\downarrow_{\zeta_1}\uparrow_{\zeta_2}\rangle \longmapsto q^{-1}|\uparrow_{\zeta_2}\downarrow_{\zeta_1}\rangle + |\downarrow_{\zeta_2}\uparrow_{\zeta_1}\rangle$$

$$q\zeta_1^{-1}|\uparrow_{\zeta_1}\downarrow_{\zeta_2}\rangle + \zeta_2^{-1}|\downarrow_{\zeta_1}\uparrow_{\zeta_2}\rangle \longmapsto q\zeta_2^{-1}|\uparrow_{\zeta_2}\downarrow_{\zeta_1}\rangle + \zeta_1^{-1}|\downarrow_{\zeta_2}\uparrow_{\zeta_1}\rangle$$

This system can be solved to get:

$$|\uparrow_{\zeta_1}\downarrow_{\zeta_2}\rangle \longmapsto \frac{-\zeta_1(q-q^{-1})}{q^{-1}\zeta_1 - q\zeta_2}|\uparrow\downarrow\rangle + \frac{(\zeta_1-\zeta_2)}{q^{-1}\zeta_1 - q\zeta_2}|\downarrow\uparrow\rangle$$

$$|\downarrow\uparrow\rangle \longmapsto \frac{\zeta_1-\zeta_2}{q^{-1}\zeta_1 - q\zeta_2}|\uparrow\downarrow\rangle - \frac{\zeta_2(q-q^{-1})}{q^{-1}\zeta_1 - q\zeta_2}|\downarrow\uparrow\rangle$$

Pole at $\zeta_1 = q^2\zeta_2$ and zero at $\zeta_1 = q^{-2}\zeta_2$.