LIE GROUPS: HOMEWORK 1

Problem 1. Let $S^2 \subset \mathbb{R}^3$ be 2–dimensional sphere (of radius 1):

$$S^2 := \{(a_1, a_2, a_3) : a_1^2 + a_2^2 + a_3^2 = 1\}$$

Describe an open covering of S^2 by open sets which are homeomorphic to open discs in $\mathbb{R}^2$ (write the explicit homeomorphism). For two open sets U and V from your cover, with $U \cap V \neq \emptyset$, let (u_1, u_2) and (v_1, v_2) be the homeomorphisms of U and V with open discs in $\mathbb{R}^2$, respectively. Pick a point $p \in U \cap V$ and write explicitly the 2×2 change of basis matrix for T_pS^2, between the following two bases:

$$\left\{ \left.\frac{\partial}{\partial u_1}\right|_p, \left.\frac{\partial}{\partial u_2}\right|_p \right\} \qquad \text{and} \qquad \left\{ \left.\frac{\partial}{\partial v_1}\right|_p, \left.\frac{\partial}{\partial v_2}\right|_p \right\}$$

Problem 2. (See Section 1.2 of Lecture 1, page 3). Recall that for a smooth m–dimensional manifold M, we introduced a topology on TM, and described an open cover $\{V_\alpha\}$ of M so that $\pi^{-1}(V_\alpha) \equiv V_\alpha \times \mathbb{R}^m$. Prove that, with this topology, $\pi : TM \to M$ is a vector bundle.

Problem 3. Let M be a smooth, m–dimensional manifold, and let $p \in M$. Consider the set of smooth paths passing through p, defined as:

$$\text{Paths}(M;p) := \{\gamma : (-\varepsilon, \varepsilon) \to M \text{ such that } \gamma \text{ is smooth, and } \gamma(0) = p\}$$

Construct the natural map $\eta : \text{Paths}(M;p) \to T_pM$, and prove that it is surjective.

Problem 4. Let G be a connected, topological group (*i.e.*, a group with a topology on its underlying set, such that multiplication and inverse are continuous).

(a) Let $H \subset G$ be an open set which is also a subgroup. Prove that H is then automatically closed.

(b) Prove that any open set $V \subset G$ containing identity $e \in V$ generates G as a group.

(c) Assuming further that G is a Lie group, prove that it admits a countable base for its topology.

Problem 5. Describle the Lie algebra of $SO_n(\mathbb{R})$.

Problem 6. Let G be a Lie group and let X be a smooth vector field on G. Prove that X is left–invariant if, and only if for every $f \in C^\infty(G)$ and $\sigma \in G$, we have

$$X \cdot (f \circ L_\sigma) = (X \cdot f) \circ L_\sigma$$

where $L_\sigma : G \to G$ is the left multiplication by $\sigma \in G$. You may use the fact that for two vector fields X_1, X_2, we have

$$X_1 = X_2 \iff X_1 \cdot f = X_2 \cdot f \text{ for every smooth function } f$$

Though, you should convince yourself that it is true!