LIE GROUPS: HOMEWORK 3

Lie algebras considered in Problems 1-3 are over $\mathbb{C}$.

Problem 1. Let $\mathbb{C}^2$ be the standard 2–dimensional representation of $\mathfrak{sl}_2$. That is, in a basis $\{|\uparrow\rangle, |\downarrow\rangle\}$, the action of the generators $\{\mathsf{e}, \mathsf{f}, \mathsf{h}\}$ is given by:

$$\begin{aligned} \mathsf{e} &: |\downarrow\rangle \mapsto |\uparrow\rangle \mapsto 0 \\ \mathsf{f} &: |\uparrow\rangle \mapsto |\downarrow\rangle \mapsto 0 \\ \mathsf{h} &: |\uparrow\rangle \mapsto |\uparrow\rangle \qquad |\downarrow\rangle \mapsto -|\downarrow\rangle \end{aligned}$$

If V_1 and V_2 are two representations of a Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ acts on $V_1 \otimes V_2$ by the following rule:

$$X \cdot (v_1 \otimes v_2) = (X \cdot v_1) \otimes v_2 + v_1 \otimes (X \cdot v_2)$$

(a) Compute the action of $\mathfrak{sl}_2$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Verify directly that $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^3 \oplus \mathbb{C}$ as $\mathfrak{sl}_2$ representations (that is, compute the basis of the two representations on the right–hand side in terms of the standard basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$).

(b) Let $V = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ terms}}$. What is the dimension of each weight space $V_{n-2k} := \{v \in V : \mathsf{h}.v = (n-2k)v\}$?

(c) Use the representation theory of $\mathfrak{sl}_2$ and part (b) above to decompose V into direct sum of irreducible representations. *Note: you do not have to compute the explicit basis of each irreducible piece.*

Problem 2. Consider the Cartan matrix of type B_2: $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. Let $\mathfrak{g} = \mathfrak{g}(\mathsf{B}_2)$ be the Lie algebra associated to this Cartan matrix.

(a) Compute the roots R of the corresponding root system. What is the dimension of $\mathfrak{g}$?

(b) Give an explicit basis of $\mathfrak{g}$ in terms of its Chevalley generators. That is, for each $\alpha \in R$, exhibit a non–zero vector in the root space $\mathfrak{g}_\alpha$ as successive brackets of the Chevalley generators. Describe the action of $\mathrm{ad}(\mathsf{f}_1)$ in terms of your basis.

Problem 3. Now let $\mathfrak{g}(\mathsf{A}_3)$ be the Lie algebra associated to the Cartan matrix of type A_3: $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Let $\{\mathsf{h}_\ell, \mathsf{e}_\ell, \mathsf{f}_\ell\}_{\ell=1,2,3}$ be Chevalley generators of $\mathfrak{g}(\mathsf{A}_3)$.

(a) Let $\sigma = (13)$ be the permutation switching 1 and 3. Prove that the following is an Lie algebra automorphism of $\mathfrak{g}(\mathsf{A}_3)$ (also denoted by σ):

$$\mathsf{e}_\ell \mapsto \mathsf{e}_{\sigma(\ell)} \qquad \mathsf{f}_\ell \mapsto \mathsf{f}_{\sigma(\ell)} \qquad \mathsf{h}_\ell \mapsto \mathsf{h}_{\sigma(\ell)}$$

(b) Let $\mathfrak{g} \subset \mathfrak{g}(\mathsf{A}_3)$ be the subspace invariant under σ. That is, $\mathfrak{g} = \{x \in \mathfrak{g}(\mathsf{A}_3) : \sigma(x) = x\}$. Prove that $\mathfrak{g}$ is a simple Lie algebra of type B_2.

Problem 4. Given 4 real numbers $\alpha, \beta, \gamma, \delta$, consider the following three dimensional Lie algebra over $\mathbb{R}$. $\mathfrak{a}$ has a basis $\{x, y, z\}$ with Lie bracket given by

$$[x, z] = \alpha x + \beta y \qquad [y, z] = \gamma x + \delta y \qquad [x, y] = 0$$

(a) Prove that $\mathfrak{a}$ is solvable.

(b) What are the relations among $\alpha, \beta, \gamma, \delta$ for rank of $\mathfrak{a}$ to be $1, 2$ or 3?

(c) Assuming $\delta = 1$, $\beta = \gamma = 0$ and $\alpha > 1$, let $\mathfrak{a}^{(\alpha)}$ be the resulting solvable Lie algebra over $\mathbb{R}$. Prove that $\{\mathfrak{a}^{(\alpha)}\}_{\alpha \in \mathbb{R}_{>1}}$ are all mutually non–isomorphic.