LIE GROUPS: HOMEWORK 4

Problem 1. Refer to Problem 2 of Homework 3. Compute a Weyl basis of the Lie algebra $\mathfrak{g}$ of type B_2. Use this to work out a compact real form of $\mathfrak{g}$.

Problem 2. Refer to Section 16.7 page 6 of Lecture 16. Prove that under the identification of quaternions $\mathbb{H} = \mathbb{C}^2$, we get $\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}(n; \mathbb{C})$.

Problem 3. Refer to Lectures 18, 19. Let $R \subset E^* \setminus \{0\}$ be a finite root system. Here E is a finite–dimensional real vector space together with the positive definite form $(.,.) : E \times E \to \mathbb{R}$. Let W be the Weyl group. Let $C_0 \subset E \setminus \cup_{\alpha \in R} \mathrm{Ker}(\alpha)$ be a fixed connected component (fundamental chamber).

(1) Prove that for $x, y \in \overline{C_0}$, if there exists $w \in W$ such that $wx = y$, then $x = y$.

(2) If in addition $x, y \in C_0$, then $w = 1$.

Problem 4. Let W be a group (not necessarily finite) generated by a finite subset of distinct elements $S = \{s_1, \cdots, s_N\} \subset W$ such that $s_i \neq 1$ and $s_i^2 = 1$ for each $1 \leq i \leq N$. As we did in class, we can define the length function on W as: for each $w \in W$, $l(w)$ is the smallest non–negative integer such that w can be expressed as a product of $l(w)$ elements of S. An expression of $w = s_{i_1} \cdots s_{i_l}$ where $l = l(w)$ is called a reduced expression of w.

Assume the following property (exchange property): for $i \in \{1, \cdots, N\}$ and $w \in W$, if $l(ws_i) < l(w)$, then for any reduced expression $w = s_{i_1} \cdots s_{i_l}$ there exists j, $1 \leq j \leq l$ such that

$$s_{i_j} \cdots s_{i_l} = s_{i_{j+1}} \cdots s_{i_l} s_i$$

For $s, s' \in S$ define $m(s, s')$ to be the order of $ss' \in W$ (possibly infinity).

Prove the following universal property: Let G be a group and let $f : S \to G$ be a set map such that $f(s)^2 = 1$ for every $s \in S$ and for every $s, s' \in S$ with $m(s, s') < \infty$, we have

$$\underbrace{f(s)f(s')f(s)\cdots}_{m(s,s')\text{ terms}} = \underbrace{f(s')f(s)f(s')\cdots}_{m(s,s')\text{ terms}}$$

Then there exists a unique group homomorphism $g : W \to G$ such that $g(s) = f(s)$ for every $s \in S$.