LIE GROUPS: HOMEWORK 5

Problem 1. Let G be a connected Lie group. Let us fix a left–invariant Haar measure on G, say $\int_G f(x)\,d\mu_l(x)$. Recall we defined the modularity function $c(t)$ on G so that

$$\int_G f(xt)\,d\mu_l(x) = c(t)\int_G f(x)\,d\mu_l(x)$$

Prove that $c(t) = \det(\mathrm{Ad}(t))$. Use this to give an alternate proof that every compact (connected) Lie group is unimodular.

Problem 2. Let G be the following Lie group (homeomorphic to $\mathbb{R}_{>0} \times \mathbb{R}$):

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, x > 0 \right\}$$

Using the previous problem, prove that G is not unimodular. Prove that the following are left and right invariant integrations on G, respectively:

$$\int f(x,y)\frac{dx}{x^2}dy \qquad \int f(x,y)\frac{dx}{x}dy$$

Problem 3. Consider the (unitary) action of the additive group $\mathbb{R}$ on $L^2(\mathbb{R})$: for each $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, we define $\pi_t f \in L^2(\mathbb{R})$ by

$$(\pi_t f)(x) = f(x - t)$$

(a) Prove that if $V \subset L^2(\mathbb{R})$ is a finite–dimensional subspace which is invariant under all π_t ($t \in \mathbb{R}$), then V contains a non–zero invariant vector.

(b) Prove that $L^2(\mathbb{R})$ has no non–zero invariant functions.

(Hence, $L^2(\mathbb{R})$ has no finite–dimensional subrepresentations of $\mathbb{R}$.)

Problem 4. Via the identifications of SU(2) (or Sp(1)) with S^3, what is the (normalized) Haar measure on these groups?