LIE GROUPS: HOMEWORK 7

For $\lambda \in P_+$, we denote by V_λ the corresponding finite–dimensional irreducible representation.

Problem 1. Prove that the character of V_{ω_1} for type G_2 is given by

$$\chi_{\omega_1} = e^{2\alpha_1+\alpha_2} + e^{\alpha_1+\alpha_2} + e^{\alpha_1} + 1 + e^{-\alpha_1} + e^{-\alpha_1-\alpha_2} + e^{-2\alpha_1-\alpha_2}$$

Problem 2. In type A_2, prove that the dimension of the irreducible representation with highest weight $m\omega_1 + n\omega_2$ is given by

$$\dim(V_{m\omega_1+n\omega_2}) = \frac{(m+1)(n+1)(m+n+2)}{2}$$

Problem 3. Prove the following decomposition in type A_3

$$V_{\omega_1} \otimes V_{\omega_1} = V_{2\omega_1} \oplus V_{\omega_2}$$

Problem 4. The Shapovalov form on the Verma module M_λ is defined to be the unique symmetric bilinear form $\mathsf{S} : M_\lambda \times M_\lambda \to \mathbb{C}$ satisfying the following conditions:

- $\mathsf{S}(\mathbf{1}_\lambda, \mathbf{1}_\lambda) = 1$
- $\mathsf{S}(e_i v, w) = \mathsf{S}(v, f_i w)$ for every $v, w \in M_\lambda$ and $i \in \{1, \cdots, l\}$.
- $\mathsf{S}(hv, w) = \mathsf{S}(v, hw)$ for every $v, w \in M_\lambda$ and $h \in \mathfrak{h}$.

Prove that the radical $\mathrm{Rad}(\mathsf{S})$ of the Shapovalov form is the unique maximal proper submodule of M_λ.

Problem 5. Let G be a simply–connected compact Lie group. Prove that every weight $\lambda \in P$ is analytic for G.