# LIE GROUPS: HOMEWORK 4 (HINT FOR PROBLEM 4)

**Problem 4.** Let $W$ be a group (not necessarily finite) generated by a finite subset of distinct elements $S = \{s_1, \cdots, s_N\} \subset W$ such that $s_i \neq 1$ and $s_i^2 = 1$ for each $1 \leq i \leq N$. As we did in class, we can define the length function on $W$ as: for each $w \in W$, $l(w)$ is the smallest non–negative integer such that $w$ can be expressed as a product of $l(w)$ elements of $S$. An expression of $w = s_{i_1} \cdots s_{i_l}$ where $l = l(w)$ is called a reduced expression of $w$.

Assume the following property (exchange property): for $i \in \{1, \cdots, N\}$ and $w \in W$, if $l(ws_i) < l(w)$, then for any reduced expression $w = s_{i_1} \cdots s_{i_l}$ there exists $j$, $1 \leq j \leq l$ such that

$$s_{i_j} \cdots s_{i_l} = s_{i_{j+1}} \cdots s_{i_l} s_i$$

For $s, s' \in S$ define $m(s, s')$ to be the order of $ss' \in W$ (possibly infinity).

Prove the following universal property: Let $G$ be a group and let $f : S \to G$ be a set map such that $f(s)^2 = 1$ for every $s \in S$ and for every $s, s' \in S$ with $m(s, s') < \infty$, we have

$$\underbrace{f(s)f(s')f(s)\cdots}_{m(s,s')\text{ terms}} = \underbrace{f(s')f(s)f(s')\cdots}_{m(s,s')\text{ terms}}$$

Then there exists a unique group homomorphism $g : W \to G$ such that $g(s) = f(s)$ for every $s \in S$.

Hint for a proof. The only non-trivial part is to prove that the following extension $g : W \to G$ is well defined. For any $w \in W$, pick a reduced expression $w = s_{i_1} \cdots s_{i_l}$ and define $g(w) = g(w; s_{i_1} \cdots s_{i_l}) := f(s_{i_1}) \cdots f(s_{i_l})$ (to emphasize the dependence on the reduced expression chosen). One needs to show that the right–hand side of this is independent of the reduced expression. Once this is done, the rest of the proof is not so hard!
To prove independence of $g(w)$ from the choice of a reduced expression, one proceeds by induction on the length of the reduced expression. The base case is trivial. For the induction step, start from two reduced expressions $w = s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_l}$. Observe that if $i_1 = j_1$ or $i_l = j_l$ the induction step will imply the result. Otherwise, apply the exchange property to the reduced expression $s_{i_1} \cdots s_{i_l}$ and $i = j_l$, to get another reduced expression

$$s_{i_1} \cdots s_{i_{t-1}} \widehat{s_{i_t}} s_{i_{t+1}} \cdots s_{i_l} s_{j_l}$$

Now this one shares the last term with $s_{j_1} \cdots s_{j_l}$ and first term with $s_{i_1} \cdots s_{i_l}$, unless $t = 1$. So, if $t \neq 1$, we are done by previous observation. The case when $t = 1$ gives $s_{i_2} \cdots s_{i_l} s_{j_l}$ as the new reduced expression. Repeated application of this argument (and also for the role of $i$'s and $j$'s flipped), will result in

$$w = \cdots s_{i_l} s_{j_l} s_{i_l} s_{j_l} = \cdots s_{j_l} s_{i_l} s_{j_l} s_{i_l}$$

at which point we are done by the hypothesis on $f$.