Lecture 0

(0.0) Overview of the course

Aim: to study structure and representation theory of Lie groups and Lie algebras

Roughly speaking - Lie group = group whose underlying set has structure of a manifold (real or complex)

Lie algebra is linearization of Lie group.

Lie Groups $\longleftrightarrow$ Lie algebras

Fundamental Theorems of Sophus Lie

$\cup$ $\qquad\qquad\qquad$ $\cup$

Compact Lie groups - - - - - - - - Semisimple Lie algebras over $\mathbb{C}$

Compact real forms

Analytic Methods

Structure (classification) Thms and repn. theory via algebraic and combinatorial methods (aka Root Systems)

(0.1) Definition. A <u>Lie group</u> G is a <u>smooth manifold</u> together with a group structure such that

Multiplication	Inverse
$G \times G \longrightarrow G$	$G \longrightarrow G$
$(g, h) \longmapsto g.h$	$g \longmapsto g^{-1}$

are <u>smooth maps</u>.

The Lie algebra of G, denoted by $\mathfrak{g} = \mathrm{Lie}(G)$, is the tangent space to G at the identity element $e \in G$.

$$\mathrm{Lie}(G) = T_e G = \text{Left-invariant vector fields on } G.$$

Remarks. Replacing "smooth manifolds" by "complex manifolds" (smooth $\rightsquigarrow$ $\mathbb{C}$-analytic), we get the notion of complex Lie group.

Similarly,

$$\left.\begin{array}{l} \text{manifold} \rightsquigarrow \text{algebraic varieties} \\ \text{smooth} \rightsquigarrow \text{morphism of algebraic varieties} \end{array}\right\} \text{algebraic groups}$$

Most generally, one can define "group object in a category".

Lie group = group object in the category of smooth manifolds.

Easy examples: $(\mathbb{R}, +)$ or more generally any real vector space.

$$S^1 = \text{circle of radius } 1 \subset \mathbb{C}^{\times} \quad (\text{non-zero complex numbers})$$

$\{ z \in \mathbb{C} : |z| = 1 \}$ subgroup

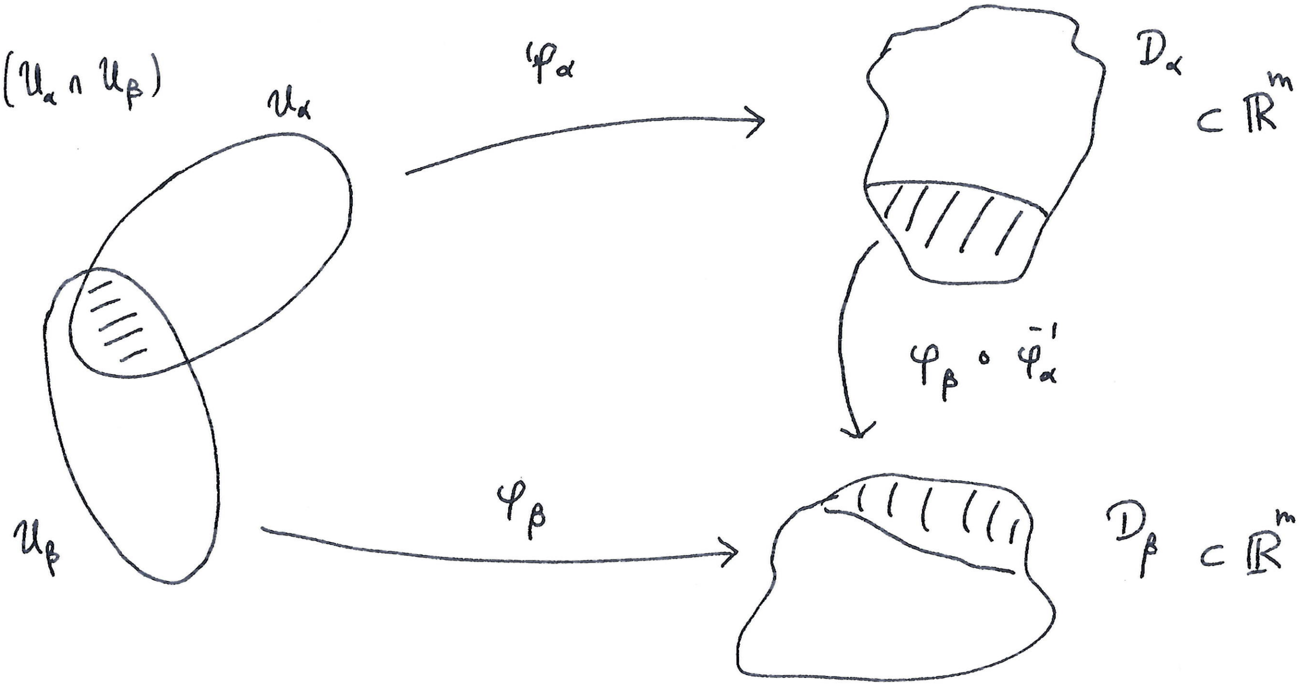
In this week's lectures, we will review basic notions of differential geometry, needed to understand this definition.

(0.2) Manifold (smooth of dimension m) M is a topological space which admits an open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ (i.e. each $\mathcal{U}_\alpha$ is open subset of M and $M = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$) such that

(i) $\forall\, \alpha \in I$, we have a homeomorphism $\varphi_\alpha : \mathcal{U}_\alpha \to D_\alpha \subset \mathbb{R}^m$, D_α is an open subset of $\mathbb{R}^m$.

(ii) $\forall\, \alpha, \beta \in I$ such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$

$$\varphi_\beta \circ \varphi_\alpha^{-1}\Big|_{\varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)} : \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \longrightarrow \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \text{ is smooth}$$



$$\varphi_\beta \circ \varphi_\alpha^{-1}\Big|_{\varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)}$$ is a map between two open subsets of $\mathbb{R}^m$

Thus it can be expressed in terms of m real valued functions $f_1(x_1, \ldots, x_m)$, $\ldots$, $f_m(x_1, \ldots, x_m)$ of m real variables.

<u>Smooth</u> means partial derivatives of f_j exist to all orders $(1 \leq j \leq m)$.

Remark. One can consider real analytic functions (Taylor series exists) instead of smooth functions to obtain the notion of real analytic manifolds. Every real analytic function is smooth but not conversely. e.g. $f(x) = e^{-1/x^2}$, $f(0) = 0$ is smooth near $x = 0$ but not real analytic.

<u>Smooth</u> map <u>between smooth manifolds M (of dim m) and N (of dim. n)</u> is a continuous map $f : M \longrightarrow N$ such that for every $p \in M$, there exists <u>coordinate neighbourhoods</u> $\mathcal{U}$ and V of p and $f(p)$ respectively such that $f(\mathcal{U}) \subset V$ and

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "f"] \arrow[d, "\text{(homeomorphism)}\ \varphi"'] & V \arrow[d, "\psi\ \text{(homeomorphism)}"] \\
D \arrow[r, dashed, "\psi \circ f \circ \varphi^{-1}"'] \arrow[d, phantom, "\cap" description] & D' \arrow[d, phantom, "\cap" description] \\
\mathbb{R}^m & \mathbb{R}^n
\end{tikzcd}$$

$\psi \circ f \circ \varphi^{-1}$ is smooth (see figure above).

Coordinate neighbourhood of a point $p \in M$ (smooth m-dim'l) is an open set $\mathcal{U} \subset M$, containing p, which is homeomorphic to an open subset of $\mathbb{R}^m$, $\varphi : \mathcal{U} \longrightarrow D \subset \mathbb{R}^m$. This φ is usually called local coordinates.

Some examples and special cases.

- A parametrized curve is a smooth map $\gamma : (a,b) \to M$ $(a<b \in \mathbb{R})$
- $C^\infty(M) := \{\text{Smooth maps } M \longrightarrow \mathbb{R}\}$: smooth functions on M.
- polynomial functions, e^x, are examples of smooth maps $\mathbb{R} \to \mathbb{R}$
- let $0 < a < b$ be real numbers. $g(t) := \begin{cases} e^{\frac{1}{(t-a)(t-b)}} & t \in (a,b) \\ 0 & \text{o/w} \end{cases}$

is a smooth function.

$$F(t) = \frac{\int_t^\infty g(s)\,ds}{\int_{-\infty}^\infty g(s)\,ds}$$ is again a smooth function

Note: $0 \leq F(t) \leq 1$ $(\forall t \in \mathbb{R})$. $F(t) = \begin{cases} 0 & \text{if } t \geq b \\ 1 & \text{if } t \leq a \end{cases}$

(0.3) Vector bundle of rank r on M is a smooth map which is surjective $\pi : E \longrightarrow M$ such that M admits an open cover $\{\mathcal{U}_j\}_{j \in J}$ satisfying :

(0.3) Tangent space T_pM to M at a point $p \in M$.

$$C^\infty(M)_p := \left\{ (f, \mathcal{U}) : \begin{array}{l} p \in \mathcal{U} \subset M \text{ is an open set} \\ f : \mathcal{U} \to \mathbb{R} \text{ is a smooth fn.} \end{array} \right\}$$

Definition : A tangent vector $L \in T_pM$ is a linear map

$$L : C^\infty(M)_p \longrightarrow \mathbb{R} \quad \text{s.t.}$$

(i) for $(f, \mathcal{U}), (g, V) \in C^\infty(M)_p$ such that $f = g$ on $\mathcal{U} \cap V$

we have $L(f) = L(g)$.

(ii) $L(f \cdot g) = L(f)\, g(p) + f(p)\, L(g)$

Because of condition (i) L descends to $C^\infty(M)_p / \sim$ where

$(f, \mathcal{U}) \sim (g, V)$ if $f = g$ on $\mathcal{U} \cap V$.

Because of condition (ii) $L(\text{constant function}) = 0$.

Proof : For $c \in \mathbb{R}$, let $\mathbb{1}_c$ denote the constant function $x \mapsto c$. $(\forall x \in M)$

Then $L(\mathbb{1}_1) = 1 \cdot L(\mathbb{1}_1) + L(\mathbb{1}_1) \cdot 1 \Rightarrow L(\mathbb{1}_1) = 0$.

$$L(\mathbb{1}_c) = L(c \cdot \mathbb{1}_1) \underset{\text{by linearity}}{=} c\, L(\mathbb{1}_1) = 0$$

□

Let $\mathfrak{m}_p = \{ [f] \in C^\infty(M)_p / \sim \; : \; f(p) = 0 \}$

Note : $\forall (f, \mathcal{U}) \in C^\infty(M)_p$, $[f - \mathbb{1}_{f(p)}] \in \mathfrak{m}_p$

and $L(f) = L([f - \mathbb{1}_{f(p)}])$.

Lemma. Every $L \in T_pM$ descends to a linear function on $\mathfrak{m}_p / \mathfrak{m}_p^2$. Let $p \in \underset{\text{open}}{\mathcal{U} \subset} M$ be a coordinate nhd. of p. Then T_pM has a basis $\left\{ \frac{\partial}{\partial u_j}\Big|_p \right\}_{j=1,\dots,m}$; $\mathfrak{m}_p / \mathfrak{m}_p^2$ has a basis $\left\{ [u_j] \text{ - denoted by } du_j|_p \right\}_{j=1,\dots,m}$ dual to each other

$$\left(\text{i.e. } \quad \frac{\partial}{\partial u_j}[u_k]\Big|_p = \delta_{jk} \right).$$

Here the homeomorphism $\varphi : \mathcal{U} \longrightarrow D \subset \mathbb{R}^m$ is denoted by $(u_1, \dots, u_m)$ - each $u_j : \mathcal{U} \longrightarrow \mathbb{R}$ is a smooth function, and translating if necessary, we are assuming that $\underline{0} \in D$, $u_j(p) = 0$.

Thus $[u_j] \in \mathfrak{m}_p / \mathfrak{m}_p^2 \quad \forall\ j = 1, \dots, m.$

Definition of $\frac{\partial}{\partial u_j}\Big|_p$:

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "{(u_1,\dots,u_m)}"] \arrow[d, squiggly] & D \subset \mathbb{R}^m \\
\mathcal{U} \cap V \arrow[r, "{\varphi = (u_1,\dots,u_m)}"] \arrow[d, "f"'] & D' \subset \mathbb{R}^m \arrow[dl, dashed, "\text{real-valued function of } m \text{ real variables, say } g(x_1,\dots,x_m)"] \\
\mathbb{R} &
\end{tikzcd}$$

Let $(f, V) \in C^\infty(M)_p$

$$\frac{\partial f}{\partial u_j}\Big|_p := \frac{\partial g}{\partial x_j}\Big|_{x_1 = \dots = x_m = 0}.$$