Lecture 1

(1.0) Sketch of a proof of Lemma (0.3) from Lecture 0, page 7.

Recall: M = smooth m-dim'l manifold. $p \in M$.

$$C^\infty(M)_p \big/ \sim \;=\; \frac{\left\{ (f, \mathcal{U}) : p \in \mathcal{U} \underset{\text{open}}{\subset} M \text{ and } f : \mathcal{U} \to \mathbb{R} \text{ is smooth} \right\}}{(f, \mathcal{U}) \sim (f', \mathcal{U}') \text{ if } f = f' \text{ on } \mathcal{U} \cap \mathcal{U}'}$$

$\cup$

$\mathfrak{m}_p$ = equivalence classes of functions vanishing at p.

$\mathfrak{m}_p^2$ = linear span of $[f \cdot g]$ where $f, g \in \mathfrak{m}_p$.
$\qquad$ ↖ equivalence class

Pick a coordinate neighbourhood $\mathcal{U} \xrightarrow[(u_1, \ldots, u_m)]{} D \subset \mathbb{R}^m$ of p. We assumed $\underline{0} = (\underbrace{0, \ldots, 0}_{m}) \in D$ and $u_i(p) = 0$ for each $i = 1, \ldots, m$.

$du_i\big|_p := [u_i] \in \mathfrak{m}_p / \mathfrak{m}_p^2$ and $\dfrac{\partial}{\partial u_i}\Big|_p \in T_p M$ is defined as

$\dfrac{\partial}{\partial u_i} f \Big|_p = \dfrac{\partial g}{\partial x_i}\Big|_{\underline{0}}$ where $(f, V) \in C^\infty(M)_p$ and

$$\begin{tikzcd} \mathcal{U} \cap V \arrow[r, "u_1, \ldots, u_m"] \arrow[d, "f"'] & D' \subset \mathbb{R}^m \arrow[dl, "g(x_1, \ldots, x_m)"] \\ \mathbb{R} & \end{tikzcd}$$

Easy part: (a) $\left\{ \dfrac{\partial}{\partial u_i}\Big|_p \right\}_{1 \le i \le m}$ and $\{du_i\}_{1 \le i \le m}$ are dual to each other.

(b) $\forall\, L \in T_p M$, $f \in \mathfrak{m}_p^2$, $L(f) = 0$ (by product rule)

We have the following lemma from Calculus (proof left as an exercise)

Lemma. Let $g(x_1, \dots, x_m)$ be a smooth function defined in an open disc $D_{\underline{0}}(r) = \{(a_1, \dots, a_m) \in \mathbb{R}^m : \sum_{j=1}^{m} a_j^2 < r\}$. If $g(\underline{0}) = 0$ and $\left.\frac{\partial g}{\partial x_i}\right|_{\underline{0}} = 0$ for $1 \leq i \leq m$, then

$$g(a_1, \dots, a_m) = \sum_{i} \sum_{j} a_i a_j \int_0^1 (1-t) \left.\frac{\partial^2 g}{\partial x_i \partial x_j}\right|_{\substack{x_1 = ta_1 \\ \vdots \\ x_m = ta_m}} . dt$$

Hence we get:

(c) if $f \in \mathfrak{m}_p$, and $\left.\frac{\partial f}{\partial u_i}\right|_p = 0$ for $1 \leq i \leq m$, then $f \in \mathfrak{m}_p^2$

(d) $\forall f \in \mathfrak{m}_p$, $[f] = \sum_{i=1}^{m} \left.\frac{\partial f}{\partial u_i}\right|_p du_i|_p$ in $\mathfrak{m}_p / \mathfrak{m}_p^2$

$\Rightarrow \quad \dim T_p M = \dim \mathfrak{m}_p / \mathfrak{m}_p^2 = m \ (= \dim M)$.

Lemma (0.3) is now proved.

$\mathfrak{m}_p / \mathfrak{m}_p^2$ is also denoted by $T_p^* M$ (cotangent space).

We will now define tangent and cotangent bundles. First we need to introduce the notion of a vector bundle.

(1.1) <u>Vector bundle (of rank r) on M</u> is the data of a smooth manifold E and a surjective smooth map $\pi : E \to M$ such that M admits an open cover $\{V_\alpha\}_{\alpha \in J}$ satisfying

(i) $\forall\, \alpha \in J$, we have homeomorphisms

$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[rr, "\sim", "\psi_\alpha"'] \arrow[dr, "\pi"'] & & V_\alpha \times \mathbb{R}^r \arrow[dl, "pr_1"] \\ & V_\alpha & \end{tikzcd}$$

s.t. $pr_1 \circ \psi_\alpha = \pi$

(ii) $\forall\, \alpha, \beta \in J$ s.t. $V_\alpha \cap V_\beta \neq \phi$, and $p \in V_\alpha \cap V_\beta$, the map $g_{\alpha\beta}(p) : \mathbb{R}^r \longrightarrow \mathbb{R}^r$ is linear iso. (here $g_{\alpha\beta}(p) \cdot v = \psi_\alpha \circ \psi_\beta^{-1}(p, v)$)

(iii)
$$\begin{aligned} g_{\alpha\beta} : V_\alpha \cap V_\beta &\longrightarrow GL_r(\mathbb{R}) \quad \text{is smooth.} \\ p &\longmapsto g_{\alpha\beta}(p) \end{aligned}$$

(1.2) <u>TM and T^*M</u>. As a set $TM = \bigcup_{p \in M} T_pM$ and

$$\begin{tikzcd} TM \arrow[d, "\pi"'] & \supset & T_pM \arrow[d] \\ M & \ni & p \end{tikzcd}$$

. There is one and only one way to put a topology on TM which makes it a vector bundle (of rank m) on M.

Idea: • Open cover needed is ~~the~~ a open cover of M by coordinate neighbourhoods $\{(\mathcal{U}_\alpha, (u_1^\alpha, \ldots, u_m^\alpha))\}_{\alpha \in I}$.

• ψ_α are given by
$$\begin{aligned} \mathcal{U}_\alpha \times \mathbb{R}^m &\longrightarrow \bigcup_{p \in \mathcal{U}_\alpha} T_pM \\ (q, (a_1, \ldots, a_m)) &\longmapsto \sum_{j=1}^m a_j \frac{\partial}{\partial u_j^\alpha}\Big|_q \in T_qM \end{aligned}$$

- put the same topology on $\bigcup_{p \in \mathcal{U}_\alpha} T_p M$ as that on $\mathcal{U}_\alpha \times \mathbb{R}^r$ via φ_α.

Remark: This idea works for T^*M or any tensor or exterior powers of these

Definition: Smooth Vector fields on M = set of smooth maps $X : M \to TM$ such that $\pi \circ X = \text{Identity}_M$. (denoted by $\text{Vect}(M)$)

1-forms on M = $\{\omega : X \to T^*M : \pi \circ \omega = \text{Identity}_M\}$ (denoted by $\Omega^1(M)$)

$\text{Vect}(M)$ acts on $C^\infty(M)$, i.e. $\forall X \in \text{Vect}(M)$, we have a linear map, also denoted by $X : C^\infty(M) \to C^\infty(M)$ defined as follows:
$\forall f \in C^\infty(M)$, $p \in M$; $(Xf)(p) = X_p(f)$ ($X_p \in T_pM$ is the value of X at p).

Leibniz rule holds: $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$.

Thus we can take composition of vector fields. However, for $X, Y \in \text{Vect}(M)$, $X \circ Y$ does not belong to $\text{Vect}(M)$.

e.g. $M = \mathbb{R}^m$, $X = \dfrac{\partial}{\partial x_i}$, $Y = \dfrac{\partial}{\partial x_j}$ $\Rightarrow X \circ Y = \dfrac{\partial^2}{\partial x_i \partial x_j}$

does not satisfy Leibniz rule.

Nevertheless, $[X, Y] := X \circ Y - Y \circ X$ is again a vector field.

(1.3) Let $f : M \to N$ be a smooth map. For every $p \in M$, we naturally get a linear map $Tf_p : T_pM \longrightarrow T_{f(p)}N$:
(or T_pf)

let $X \in T_pM$,
$(g, V) \in C^\infty(N)_{f(p)}$ (i.e. $V \subset N$ is an open set containing $f(p)$ and $g : V \to \mathbb{R}$ is a smooth function)

Then $\left(T_pf\,(X)\right)(g) = X(g \circ f)$

$$g \circ f : f^{-1}(V) \xrightarrow{f} V \xrightarrow{g} \mathbb{R} \quad \in C^\infty(M)_p$$

(again open in M)

$\Longrightarrow$ We get a smooth map $Tf : TM \longrightarrow TN$ such that

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd} \qquad \pi_N \circ Tf = f \circ \pi_M.$$

(1.4) Back to Lie groups and Lie algebras.

G = Lie group (manifold + group s.t. $G \times G \xrightarrow{\text{Multiplication}} G$, $G \xrightarrow{\text{inverse}} G$ are smooth)

$\forall g \in G$, $l_g : G \longrightarrow G$, $h \longrightarrow gh$ is a diffeomorphism (smooth bijection with smooth inverse $= l_{g^{-1}}$)

left-translation

Similarly we have r_g (right-translation) ; $\text{Conjugation}_g(x) = gxg^{-1}$

Definition. A vector field $X \in \mathrm{Vect}(G)$ is said to be <u>left-invariant</u> if $(T\ell_g)(X(h)) = X(gh)$ $\forall\, g, h \in G$.

$$\begin{array}{ccc} T_h G & \xrightarrow{\sim} & T_{gh} G \\ & \uparrow & \\ G & \xrightarrow[\ell_g]{\text{diffeo.}} & G \\ \Psi & & \Psi \\ h & \longmapsto & gh \end{array}$$

$\mathrm{Vect}(G)^G$ = space of left-invariant vector fields

Easy check: $X, Y \in \mathrm{Vect}(G)^G$
$\Rightarrow [X,Y] \in \mathrm{Vect}(G)^G$

Clearly any $X \in \mathrm{Vect}(G)^G$ is determined by its value at $e \in G$ since $X(g) = (T\ell_g)(X(e))$.

Conversely, given $L \in T_e G$, define $X \in \mathrm{Vect}(G)$ by
$$X(g) = (T\ell_g)(L).$$

$X \in \mathrm{Vect}(G)^G$: $\quad X(gh) = (T\ell_{gh})(L) = (T\ell_g \circ T\ell_h)(L)$
$$= (T\ell_g)(X(h)) \qquad (\forall\, g, h \in G).$$

<u>Prop.</u> $\mathfrak{g} = T_e G \simeq \mathrm{Vect}(G)^G$. Thus we have an operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$. (called Lie bracket).

(Skew-symmetry) $\quad [X,Y] + [Y,X] = 0$

(Jacobi Identity) $\quad [X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]].$

(1.5) Definition. A Lie algebra $\mathfrak{a}$ (over a field - $\mathbb{R}$ or $\mathbb{C}$ for us) is a vector space, together with a bilinear map

$$[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$$

satisfying skew-symmetry and Jacobi identity.

Summarizing:

$$G : \text{Lie group} \Longrightarrow \mathfrak{g} = T_eG \underset{\text{Lie}(G)}{=} : \text{Lie algebra}$$

$$f : G \longrightarrow H \Longrightarrow \text{Lie}(f) = T_e f : \mathfrak{g} \longrightarrow \mathfrak{h}$$

Caution: a smooth map $f : M \to N$ of manifolds does <u>not</u> induce a map between $\text{Vect}(M) \longrightarrow \text{Vect}(N)$. Only $Tf : TM \longrightarrow TN$. For Lie groups however, $\text{Lie}(f)$ is a map $\text{Vect}(G)^G \longrightarrow \text{Vect}(H)^H$ (here $f : G \longrightarrow H$).