Lecture 2

(2.0) Recall: G = a Lie group. $\mathfrak{g} = \mathrm{Lie}(G) := T_e G \simeq \mathrm{Vect}(G)^G$

Remark (problem from Homework 1). For $f \in C^\infty(G)$, $g \in G$, we have

$f \circ l_g \in C^\infty(G)$. ($l_g : G \to G$ is left multiplication by g)

Then $X \in \mathrm{Vect}(G)^G \iff (X \cdot f) \circ l_g = X \cdot (f \circ l_g)$.

(2.1) Some examples of Lie groups.

$GL_n(\mathbb{R})$ = $n \times n$ invertible matrices with entries from $\mathbb{R}$

$$= \left\{ A = (a_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n^2} : \det(A) \neq 0 \right\} \underset{\text{open}}{\subset} \mathbb{R}^{n^2}$$

is a Lie group (of dimension n^2).

$\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}_{\neq 0}$ is a continuous map. Since $\mathbb{R}_{\neq 0}$ is not connected, $GL_n(\mathbb{R})$ has two connected components: $\det > 0$; $\det < 0$.

$SL_n(\mathbb{R}) \ni A$ iff $\det(A) = 1$ is a closed subgroup of $GL_n(\mathbb{R})$.

$O_n(\mathbb{R})$: let $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a positive-definite symmetric bilinear form. $A \in O_n(\mathbb{R})$ iff $(Av, Aw) = (v, w)$ for every $v, w \in \mathbb{R}^n$.

$SO_n(\mathbb{R}) \subset O_n(\mathbb{R})$ given by $\det = 1$.

(2.2) A trick to compute Lie algebra of a Lie group $G \underset{\text{closed}}{\subset} GL_n$. ②

Assume G is defined by equations

$$f_1(\underline{x}) = f_2(\underline{x}) = \dots = f_r(\underline{x}) = 0 \quad , \text{ where } \underline{x} = (x_{ij} : 1 \le i,j \le n)$$

and each f_j is a (real-valued) function of n^2 variables.

Set $x_{ij} = \delta_{ij} + \varepsilon\, y_{ij}$ (or $(x_{ij})_{1 \le i,j \le n} = \mathrm{Id}_{n\times n} + \varepsilon \cdot (y_{ij})_{1 \le i,j \le n}$)

For each $\ell = 1, \dots, r$ we can compute

$$f_\ell\left((\delta_{ij} + \varepsilon\, y_{ij})_{1 \le i,j \le n} \right) = \underbrace{f_\ell((\delta_{ij}))}_{=0} + \varepsilon\, g_\ell((y_{ij})) + \cdots$$

$= 0$ because we are assuming $\mathrm{Id}_{n\times n} \in G$.

$\mathrm{Lie}(G) \subset$ Space of $n\times n$ matrices is then given by

$$g_1(\underline{y}) = \dots = g_r(\underline{y}) = 0.$$

e.g. (easy) $G = S^1 \subset \mathbb{R}^2 \setminus \{(0,0)\}$ with $\cdot$ given by identifying it with $\mathbb{C}\setminus\{0\}$.

$S^1 = (x_1^2 + x_2^2 - 1 = 0.)$

$e = (1,0)$. Put $(x_1, x_2) = e + \varepsilon(y_1, y_2) = (1 + \varepsilon y_1, \varepsilon y_2)$

$$x_1^2 + x_2^2 - 1 = (1+\varepsilon y_1)^2 + (\varepsilon y_2)^2 - 1 = \varepsilon(\underbrace{2y_1}_{g}) + \varepsilon^2(y_1^2 + y_2^2)$$

$T_e S^1 = \{y_1 = 0\}$ vertical line as we expect.

Another example $G = SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$. is defined by

$$f(X) = \det(X) - 1 = \sum_{\pi \in S_n} (-1)^{\mathrm{sgn}(\pi)} x_{1,\pi(1)} \cdots x_{n,\pi(n)} - 1 = 0$$

$$f(\mathrm{Id} + \varepsilon Y) = \sum_{\pi \in S_n} (-1)^{\mathrm{sgn}(\pi)} \left(\delta_{1,\pi(1)} + \varepsilon y_{1,\pi(1)}\right) \cdots \left(\delta_{n,\pi(n)} + \varepsilon y_{n,\pi(n)}\right) - 1$$

If $\pi \neq$ Identity permutation, at least two $a, b \in \{1, \dots, n\}$ get $a \neq \pi(a)$ & $b \neq \pi(b)$

corresponding term will have ε^2 in it.

$\pi =$ Identity gives $\quad 1 + \varepsilon(y_{11} + \dots + y_{nn}) + \varepsilon^2(\cdots) - 1$

$g(Y) = \mathrm{Tr}(Y)$

So $\mathrm{Lie}(SL_n(\mathbb{R}))$ is given by matrices of trace 0.

(denoted by $\mathfrak{sl}_n(\mathbb{R})$)

Alternately, $\det(\mathrm{Id} + \varepsilon Y) = 1 + \varepsilon \cdot \mathrm{Tr}(Y) + \dots + \varepsilon^n \det(Y)$

"Characteristic poly."

Really easy : $G = GL_n(\mathbb{R})$, $T_e G = \mathrm{Lie}(G) =$ all $n \times n$ matrices.

denoted by $\mathfrak{gl}_n(\mathbb{R})$

(2.3) A <u>homomorphism of Lie groups</u> $\varphi : G \longrightarrow H$ is a group homomorphism which is a smooth map of manifolds.

We say <u>G' is (Lie) subgroup of G</u> if we ~~ea~~ have an injective homomorphism $G' \overset{\varphi}{\hookrightarrow} G$ s.t. $\forall\ x \in G'$,

$$T_x\varphi : T_x G' \longrightarrow T_{\varphi(x)} G \text{ is also injective.}$$

Important remark: G' and $\varphi(G')$ need not be homeomorphic as topological spaces

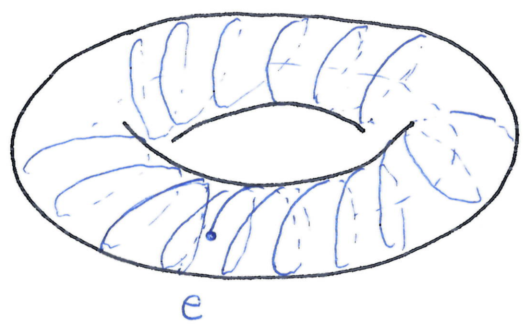
$\boxed{\text{with subspace topology as } \varphi(G') \subset G}$

e.g. $T_2 := S^1 \times S^1$ (torus) Let $\mathbb{R} \longrightarrow T$ be irrational line

for instance, $T_2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ and

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & T \\ \Psi & & \Psi \\ a & \longmapsto & (a, \alpha a) \bmod \mathbb{Z}^2 \end{array}$$

α irrational



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A <u>homomorphism of Lie algebras</u> $\xi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is a linear map (of vector spaces) s.t. $[\xi(X), \xi(Y)] = \xi([X,Y])\ \forall X, Y \in \mathfrak{g}$

$\mathfrak{h} \subset \mathfrak{g}$ is a <u>Lie subalgebra</u> if $X, Y \in \mathfrak{h} \Rightarrow [X,Y] \in \mathfrak{h}$

subspace

$\mathfrak{h} \subset \mathfrak{g}$ is an <u>ideal</u> if $X \in \mathfrak{g},\ Y \in \mathfrak{h} \Rightarrow [X,Y] \in \mathfrak{h}$.

(2.4) Lie's Theorems - informal version. We can view these statements as a dictionary between Lie groups and Lie algebras

G = Lie group $\rightsquigarrow$ $\mathfrak{g} = T_eG = \mathrm{Vect}(G)^G$ Lie algebra

$H \hookrightarrow G$ subgroup $\rightsquigarrow$ $\mathfrak{h} \subset \mathfrak{g}$ subalgebra

Normal subgroup $\rightsquigarrow$ ideal

Hom of Lie groups $\varphi : G \to G'$ $\rightsquigarrow$ Hom of Lie algebras $\underbrace{T_e\varphi}_{= \mathrm{Lie}(\varphi)} : \mathfrak{g} \to \mathfrak{g}'$

And the converse holds (mostly).

More precisely, let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then there is a Lie subgroup $H \hookrightarrow G$ s.t. $\mathfrak{h} = \mathrm{Lie}(H)$.

If $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}' = \mathrm{Lie}(G')$ and $\xi : \mathfrak{g} \to \mathfrak{g}'$ is a hom. of Lie algebras, and assuming <u>G is simply-connected</u>; we have a hom. of Lie groups $\varphi : G \to G'$ s.t. $\xi = \mathrm{Lie}(\varphi)$

Remark. We are working with connected Lie groups here. Since T_eG only cares about connected component of G containing e. In the last assertion, the hypothesis of G being simply-connected cannot be removed

e.g.

$$\begin{array}{ccc} \mathrm{Lie}(S^1) & & \mathrm{Lie}(\mathbb{R}) \\ \| & & \| \\ \mathbb{R} & \xrightarrow[\mathrm{Id}]{} & \mathbb{R} \end{array}$$

cannot be lifted, as there is no ~~of~~ hom. from S^1 to $\mathbb{R}$.