Lecture 3

(3.0) Recall : we have G : Lie group $\rightsquigarrow$ $\mathfrak{g} = \mathrm{Lie}(G)$ Lie algebra.

And we are aiming to prove the following

Theorem : (1) For every hom. of Lie groups $\varphi : G \to G'$, the induced map $\mathrm{Lie}(\varphi) = T_e\varphi : \mathrm{Lie}(G) \to \mathrm{Lie}(G')$ is a hom. of Lie algebras.

[Hence we have a functor Lie : Lie Groups $\longrightarrow$ Lie Algebras.]

(2) Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ be a subalgebra. Then $\exists!$ Lie subgroup $\varphi : H \hookrightarrow G$ s.t. $\mathfrak{h} = \mathrm{Lie}(H)$ and $i = \mathrm{Lie}(\varphi)$.

(3) Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}' = \mathrm{Lie}(G')$. Assume G is connected and simply-connected. Let $\xi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ be a hom of Lie alg. Then $\exists!$ $\varphi : G \to G'$ hom. of Lie groups, s.t. $\xi = \mathrm{Lie}(\varphi)$.

(3.1) Proof of Theorem (3.0) (1) : We have $G \xrightarrow{\varphi} G'$ inducing

$$\begin{tikzcd}
L \in \mathfrak{g} = T_eG \arrow[r, "T_e\varphi =: \xi"] \arrow[d, equal] & T_{e'}G' = \mathfrak{g}' \ni L' = T_e\varphi(L) \arrow[d, equal] \\
X \in \mathrm{Vect}(G)^G \arrow[r] & \mathrm{Vect}(G')^{G'} \ni X'
\end{tikzcd}$$

with $L \leftrightarrow X$ and $L' \leftrightarrow X'$.

To prove : $[\xi(L_1), \xi(L_2)] = \xi([L_1, L_2])$ $\forall\, L_1, L_2 \in \mathfrak{g}$.

Claim : with the notations of the picture above, $\forall\, f \in C^\infty(G')$

$$X \cdot (f \circ \varphi) = (X' \cdot f) \circ \varphi \quad \left(\in C^\infty(G)\right)$$

Assuming the claim, let $L_1, L_2 \in \mathfrak{g} = T_e G$ and use the same notations as in the picture above. (i.e. $X_1, X_2 \in Vect(G)^G$ corr. to $L_1, L_2 \in T_e G$; $L'_\ell = T_e\varphi(L_\ell)$ $(\ell = 1, 2)$; $X'_1, X'_2 \in Vect(G')^{G'}$ corr. to $L'_1, L'_2 \in T_{e'} G'$). Then $\forall f \in C^\infty(G')$ we have

$$X_1 \cdot (X_2 \cdot (f \circ \varphi)) = X_1 \cdot ((X'_2 \cdot f) \circ \varphi) = (X'_1 \cdot (X'_2 \cdot f)) \circ \varphi$$

$$\Rightarrow \quad [X_1, X_2] \cdot (f \circ \varphi) = ([X'_1, X'_2] \cdot f) \circ \varphi \quad \in C^\infty(G)$$

Evaluate at $e \in G$ to get $[X_1, X_2]_e \cdot (f \circ \varphi) = [X'_1, X'_2]_{e'} \cdot f$

$$\xi([L_1, L_2]) \cdot f = T_e\varphi([L_1, L_2]) \cdot f = [L'_1, L'_2] \cdot f \quad (\forall f \in C^\infty(G'))$$

Hence $\xi([L_1, L_2]) = [\xi(L_1), \xi(L_2)]$

<u>Proof of Claim</u>: Let $\sigma \in G$. Then

$$(X \cdot (f \circ \varphi))(\sigma) = X_\sigma \cdot (f \circ \varphi) = (T_e \ell_\sigma (L)) \cdot (f \circ \varphi)$$

$$= ((T_\sigma \varphi \circ T_e \ell_\sigma)(L)) \cdot f$$

$$= (T_e (\varphi \circ \ell_\sigma))(L) \cdot f = (T_e (\ell_{\varphi(\sigma)} \circ \varphi))(L) \cdot f$$

φ is a group hom

$$= (T_{e'} \ell_{\varphi(\sigma)} \circ T_e \varphi)(L) \cdot f$$

$$= (T_{e'} \ell_{\varphi(\sigma)})(L') \cdot f = X'_{\varphi(\sigma)} \cdot f$$

$$= (X' \cdot f)(\varphi(\sigma)) \quad ✓$$

(recall: ℓ_σ = left mult. by σ)

(3.2) Ingredients for (2) and (3) of Theorem (3.0). ③

For (2): we are given $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Vect}(G)^G$ and we want to construct a manifold $H \hookrightarrow G$ s.t. $T_\sigma H = \{X_\sigma : X \in \mathfrak{h}\}$

This is the content of <u>Frobenius' Theorem on Integral manifolds</u>.

For (3):

$$\mathfrak{g} \xrightarrow{\xi} \mathfrak{g}' \quad \rightsquigarrow \quad e \in V \xrightarrow{\bar{\varphi}} G', \quad \underset{\text{open}}{V \subset G}$$

and extend $\bar{\varphi}$ to entire G since V generates G. The obstruction to this extension is precisely in $\pi_1(G)$ (Monodromy principle).

(3.3) Statement of Frobenius Thm.

(Hypotheses) M: an m-dim'l smooth manifold

$\mathcal{X}(p) \subseteq T_p M$ an n-dim'l subspace $\forall\, p \in M$ s.t.

(1) $\mathcal{X}$ is smooth, i.e. $\forall\, p \in M$, $\exists\, \mathcal{U} \underset{\text{open}}{\subset} M$, $p \in \mathcal{U}$ ~~s.t.~~ and $X_1, \dots, X_n$ smooth vector fields on $\mathcal{U}$ s.t. $\{X_1(q), \dots, X_n(q)\}$ form a basis of $\mathcal{X}(q)$ $\forall\, q \in \mathcal{U}$

(2) $\mathcal{X}$ is involutive i.e. if $V \subset M$ is open, X, Y are vector fields on V s.t. $X(q), Y(q) \in \mathcal{X}(q)$ $\forall q \in V$

then $[X,Y](q) \in \mathcal{X}(q)$ $\forall q \in V$.

Conclusion: (Local statement) $\forall p \in M$, $\exists$ a coord. nhd. of p

$$(u_1, \dots, u_m) : \mathcal{U} \longrightarrow \mathrm{Cube}_r(\underline{0}) := \{(a_1, \dots, a_m) \in \mathbb{R}^m : |a_j| < r\}_{j=1,\dots,m}$$

s.t. (i) $u_j(p) = 0$ for $j = 1, \dots, m$

(ii) for any $b_{n+1}, \dots, b_m \in (-r, r)$, define

$$S_{\underline{b}} = \{q \in \mathcal{U} : u_\ell(q) = b_\ell \ \forall\, n+1 \le \ell \le m\}. \text{ Then}$$

$$T_q S_{\underline{b}} = \mathcal{X}(q) \quad \forall q \in S_{\underline{b}}$$

w.r.t. $\mathcal{X}$

A submanifold $W \xrightarrow{\varphi} M$ is said to be <u>integral</u> if

$$T_p\varphi\,(T_pW) = \mathcal{X}(p) \subset T_pM, \ \forall p \in W.$$

(recall: submanifold $\equiv$ φ is inj. and $T_p\varphi$ is injective $\forall p$)

(Global Statement) $\forall p \in M$, $\exists!$ max'l integral submanifold $W(p)$ of M. Any integral submanifold S of M s.t. $p \in S$ is an open submanifold of $W(p)$.