Lecture 4

(4.0) Recall: we need to prove Frobenius Thm and Monodromy principle in order to finish the proof of Lie's Theorems.

Proof of Frobenius Theorem is by induction. We begin by proving the local version (see (3.3) of Lecture 3, page 3).

(4.1) Let M be a smooth m-dim'l manifold, $p \in M$ and let X be a smooth vector field defined in an open set containing p, s.t. $X_p \neq 0$. Then we can find a coordinate nhd. $(u_1, \dots, u_m) : \mathcal{U} \longrightarrow \text{Cube}_r(\underline{0}) \subset \mathbb{R}^m$ s.t. (a) $u_j(p) = 0 \quad \forall j = 1, \dots, m$ (b) X is defined on $\mathcal{U}$ and $X = \frac{\partial}{\partial u_1}$ on $\mathcal{U}$.

Proof. Let us start with any coordinate nhd. of p where X is defined:

$$
\begin{array}{ccccc}
(z_1, \dots, z_m) : & W & \longrightarrow & \text{Cube}_t(\underline{0}) & \subset \mathbb{R}^m \\
& \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\
& p & \longmapsto & (0, \dots, 0) &
\end{array}
$$

s.t. $X_p z_1 \neq 0$. We want to construct a change of coordinates

$\underline{\varphi} = (\varphi_1(a_1, \dots, a_m), \dots, \varphi_m(a_1, \dots, a_m))$ on $\text{Cube}_r(\underline{0}) \subset \mathbb{R}^m$ s.t.

(for some $r < t$)

$$
\begin{array}{ccc}
W & \xrightarrow[\text{Given}]{(z_1, \dots, z_m)} & \text{Cube}_t(\underline{0}) \\
\uparrow & & \uparrow \ \underline{\varphi} = (\varphi_1, \dots, \varphi_m) \leftarrow \text{To construct} \\
\mathcal{U} & \xrightarrow[(u_1, \dots, u_m)]{} & \text{Cube}_r(\underline{0})
\end{array}
$$

$(u_1, \dots, u_m)$ (define using $\underline{z}$ and $\underline{\varphi}$)

and $X = \frac{\partial}{\partial u_1}$ on $\mathcal{U}$.

Let $F_j(b_1, \dots, b_m)$ be real valued functions on m real variables $(1 \le j \le m)$ $(b_1, \dots, b_m) \in \text{Cube}_t(\underline{0})$ defined by:

$$F_j(z_1(q), \dots, z_m(q)) = X_q z_j \qquad \forall\, q \in W.$$

Claim: If $\dfrac{\partial \varphi_j}{\partial a_1} = F_j(\varphi_1, \dots, \varphi_m)$ $\quad (1 \le j \le m)$, with initial conditions $\varphi_1(0, a_2, \dots, a_m) = 0$, $\quad \varphi_i(0, a_2, \dots, a_m) = a_i \quad (2 \le i \le m)$

then $X = \dfrac{\partial}{\partial u_1}$.

For $q \in \mathcal{U}$ and $\underline{b} = (b_1, \dots, b_m) = (u_1(q), \dots, u_m(q))$ we have

$$\frac{\partial \varphi_j}{\partial a_1}(\underline{b}) = F_j(\varphi_1(\underline{b}), \dots \varphi_m(\underline{b})) = F_j(z_1(q), \dots, z_m(q))$$
$$= X z_j(q)$$

$$\Rightarrow \quad X z_j(q) = \frac{\partial \varphi_j}{\partial a_1}(u_1(q), \dots, u_m(q)) = \left.\frac{\partial z_j}{\partial u_1}\right|_q \quad \text{as claimed.}$$

We still have to show that $(\varphi_1, \dots, \varphi_m)$ is a change of coordinates.

i.e. $\det\left[\dfrac{\partial \varphi_i}{\partial a_j}\right] \neq 0$ at $(0, \dots, 0)$. $\quad \dfrac{\partial \varphi_1}{\partial a_1}(\underline{0}) = F_1(\underline{0}) = X_p z_1$

$\neq 0$ by assumption

$\dfrac{\partial \varphi_i}{\partial a_j}(\underline{0}) = \delta_{ij} \quad (2 \le i, j \le m)$ by initial conditions. So

$$\left[\frac{\partial \varphi_i}{\partial a_j}\right]_{1 \le i,j \le m} = \begin{bmatrix} F_1(\underline{0}) & * & \cdots & * \\ * & & & \\ \vdots & & \mathrm{Id}_{m-1 \times m-1} & \\ * & & & \end{bmatrix} \rightsquigarrow \det = F_1(\underline{0}) \neq 0.$$

$\square$

(4.2) Induction step. Now we have n smooth vector fields, $X_1, \dots, X_n$ defined on an open set W containing p s.t. $\{X_1(q), \dots X_n(q)\}$ form a basis of $\mathcal{X}(q)$ $(\forall q \in W)$ and $[X_i, X_j] = \sum_{k=1}^{n} g_{ij}^{k} X_k$ for some smooth fns. g_{ij}^k on W.

Pick a coordinate system around p, say $Z \subset W$, with $(z_1, \dots, z_m) : Z \to \mathrm{Cube}_s(\underline{0}) \subset \mathbb{R}^m$ s.t. $X_1 = \frac{\partial}{\partial z_1}$. Replace X_j $(2 \leq j \leq n)$ by $X_j - (X_j z_1) X_1$ so that $X_j z_1 = 0$ $\forall j = 2, \dots, n$.

$\overline{Z} = \{q \in Z : z_1(q) = 0\}$ has $(n-1)$ smooth vector fields $X_2, \dots, X_n$ with the same properties. By induction, choose coordinate system around $p \in \overline{Z}$, $(w_2, \dots, w_m) : W \to \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^{m-1}$ s.t. the statement of theorem holds: i.e. $\forall$ $b_{n+1}, \dots, b_m \in (-r, r)$, the slice $\overline{S}_b \subset W$ defined by $w_\ell(q) = b_\ell$ $(n+1 \leq \ell \leq m)$ has the property $T_q \overline{S}_b =$ span of $\{X_2(q), \dots, X_n(q)\}$ $\forall q \in \overline{S}_b$.

$$\begin{array}{ccc} Z & \xrightarrow{(z_1, \dots, z_m)} & \mathrm{Cube}_s(\underline{0}) \subset \mathbb{R}^m \\ \uparrow & & \uparrow \\ \{z_1 = 0\} \supset W & \xrightarrow[(w_2, \dots, w_m)]{} & \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^{m-1} \end{array}$$

slice of Z

Take $\mathcal{U}$ to be open subset in Z defined by

$q \in \mathcal{U} \iff (0, z_2(q), \ldots, z_m(q)) \in \mathrm{Cube}_r(\underline{0}) \hookrightarrow \mathrm{Cube}_s(\underline{0})$ and

$|z_1(q)| < r$

Coordinates $u_1 = z_1, \; u_2 = w_2, \; \ldots, \; u_m = w_m$.

Claim: $X_i u_\ell = 0 \quad \forall \; 1 \leq i \leq n$ and $n+1 \leq \ell \leq m$.

True for $i=1$ by construction. For $i \geq 2$, we have

$$\frac{\partial}{\partial u_1}(X_i u_\ell) = X_1 X_i u_\ell = [X_1 X_i] u_\ell = \sum_{j=2}^{n} g_{1i}^{j} X_j u_\ell$$

Since on $u_1 = 0$ the functions $X_i u_\ell$ vanish, this eq^n implies that the fns. $\psi_i = X_i u_\ell$ are solns. of homogeneous diff'l eq^{n}s $(2 \leq i \leq n)$

$$\frac{\partial \psi_i}{\partial u_1} = \sum_{j=2}^{n} g_{1i}^{j} \psi_j$$ w/ initial condition $\psi_i = 0$ $2 \leq i \leq n$ (on $u_1 = 0$)

By uniqueness, $\psi_j = 0 \quad \forall \; 2 \leq j \leq n$ and we are done.

$\square$

(4.3) Global statement.

We define a new topological space $M(\mathcal{X})$ which is same as M (as a set) with a base of topology given by

$\mathcal{B} = \{ S \hookrightarrow M$ s.t. S is an integral submanifold w.r.t. $\mathcal{X} \}$

That is, a subset $O \subset M(\mathcal{X})$ is open iff it is a union of integral submanifolds.

The local statement can be used to prove that

- $\mathcal{B}$ is a base of a topology on $M(\mathcal{X})$ (i.e. $\bigcup_{S \in \mathcal{B}} S = M$

and $\forall\ S_1, S_2 \in \mathcal{B}$, $p \in S_1 \cap S_2$, $\exists\ S_3 \in \mathcal{B}$ s.t. $p \in S_3 \subset S_1 \cap S_2$)

- $M(\mathcal{X})$ is a manifold.

Then $\forall\ p \in M$, the connected component of p in $M(\mathcal{X})$ is the maximal integral submanifold $W(p)$ claimed in (Global Statement) of Frobenius Theorem (Thm (3.3) of Lecture 3 page 4).

(4.4) Proof of Lie's Theorem (Thm (3.0) of Lecture 3, part (2)).

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ be a subalgebra. Define

$$\mathcal{X}_{\mathfrak{h}}(\sigma) := \left\{ X(\sigma) \ : \ X \in \mathfrak{h} \subset \mathfrak{g} = \mathrm{Vect}(G)^G \right\} \subset T_\sigma G.$$

$\mathcal{X}_{\mathfrak{h}}$ satisfies the hypotheses of Frobenius' Thm. (being a subalgebra $\equiv$ $\mathcal{X}_{\mathfrak{h}}$ is involutive).

Let $H \overset{\varphi}{\hookrightarrow} G$ be the max'l integral submanifold passing through $e \in G$ (in the notation of Frobenius' Thm $H = W(e)$).

Claim: H is a subgroup.

Since $\underset{\substack{\| \\ \text{left-inv. vector fields}}}{\mathfrak{h} \subset \mathfrak{g}}$, $\forall\ \sigma \in G$, l_σ leaves $\mathcal{X}_{\mathfrak{h}}$ invariant i.e. l_σ permutes the max'l integral submanifolds of $\mathcal{H}_{\mathfrak{h}}$.

Now let $\sigma \in H$. Then $l_{\sigma^{-1}}(H)$ is the max'l integral submanifold and $e \in l_{\sigma^{-1}}(H) \Rightarrow l_{\sigma^{-1}}(H) = H$

So $\sigma^{-1} \in H$ (since $e \in H$) and H is stable under left multiplication.

$\Rightarrow$ H is a subgroup of G.

Now consider

$$\begin{tikzcd} H \times H \arrow[r, hook, "\varphi \times \varphi"] \arrow[dr, dashed] & G \times G \arrow[r] & G \\ & {[(\sigma,\tau)} \arrow[r, mapsto] & \sigma^{-1}\tau] \\ & H \arrow[ur, hook, "\varphi"'] & \end{tikzcd}$$

since image of the horizontal map lands inside $\varphi(H)$

The fact that the resulting map $\begin{array}{c} H \times H \longrightarrow H \\ (\sigma, \tau) \longmapsto \sigma^{-1}\tau \end{array}$ is smooth, is a consequence of countability axiom (from HW1 (problem 4(c))). (See Optional Reading A for a proof).

$\square$

(4.5) Proof of Lie's Theorem part (3).

Set up: $\mathfrak{g} = \mathrm{Lie}(G)$ when G is connected & simply-conn.

$\mathfrak{g}' = \mathrm{Lie}(G')$

$\xi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ hom of Lie algebras.

To prove: $\exists!\ \varphi : G \to G'$ s.t. $\xi = T_e\varphi$ (or $\mathrm{Lie}(\varphi)$)

(Existence) Let $\mathfrak{a} = \{(x, \xi(x)) : x \in \mathfrak{g}\} \subset \mathfrak{g} \times \mathfrak{g}'$

Subalgebra since ξ is hom. of Lie algebras

By (4.4) we get a Lie subgroup

$$\begin{array}{ccccc} A & \xhookrightarrow{\psi} & G \times G' & & \\ & \swarrow^{pr_1} & & \searrow^{pr_2} & \\ G & & & & G' \end{array}$$

$\psi_1 = pr_1 \circ \psi : A \longrightarrow G$

Note: $\mathrm{Lie}(\psi_1) = \mathfrak{a} \subset \mathfrak{g} \times \mathfrak{g}' \xrightarrow{pr_1} \mathfrak{g}$, $\mathfrak{a} \xrightarrow{\sim} \mathfrak{g}$

$\Rightarrow \exists$ a nhd. $V \underset{\text{open}}{\subset} G$ of e s.t. ψ_1 admits an inverse $\lambda : V \longrightarrow A$. Clearly $\lambda(e) = e \in A$ and if $a, b, ab \in V$ then $\lambda(ab) = \lambda(a)\lambda(b)$.

G simply-connected $\Rightarrow \lambda$ extends to a unique hom (☆) $G \xrightarrow{\lambda} A$. Let $\varphi = pr_2 \circ \lambda : G \to G'$. By construction $\mathrm{Lie}(\varphi) = \xi$.

(Uniqueness) $\mathrm{Lie}(\text{graph of } \varphi) \subset \mathrm{Lie}(G \times G') = \mathfrak{g} \times \mathfrak{g}'$ depends only on ξ. Hence, graph of φ, and therefore φ, is uniquely determined by ξ.