Lecture 5

(5.0) Today we are going to discuss some topological constructions. For us a topological space will be assumed to be Hausdorff and locally (path) connected.

Defn. (Covering space) $\begin{array}{c} E \\ \pi\downarrow \\ X \end{array}$ is a covering space if $\forall\, x \in X$, there exists an open set $\mathcal{U} \subset X$, $x \in \mathcal{U}$ s.t. (1) $\pi^{-1}(\mathcal{U})$ is a disjoint union of open sets in E

$\pi^{-1}(\mathcal{U}) = \bigsqcup_{\alpha \in A} \tilde{\mathcal{U}}_\alpha$ (2) $\pi\big|_{\tilde{\mathcal{U}}_\alpha} : \tilde{\mathcal{U}}_\alpha \to \mathcal{U}$ is a homeomorphism.

We say X is simply-connected if X is connected and if $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ is a connected covering space then $E \xrightarrow{\pi} X$ is a homeomorphism.

We define $\pi_1(X)$ to be the group of homeomorphisms $\begin{tikzcd} \tilde{X} \arrow[rr, "\eta", "\simeq"'] \arrow[dr, "\pi"'] & & \tilde{X} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$ s.t. $\eta \circ \pi = \pi$; where $\tilde{X}$ is a simply-connected covering of X.

Facts from topology: (1) X admits a simply-connected covering space, unique up to iso, $\tilde{X}$ - called universal covering space of X (assuming X is connected).

(2) If $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ is a covering space, and $E_0 \subset E$ is a connected component, then $\begin{array}{c} E_0 \\ \downarrow \\ X \end{array}$ is again a covering space

(3) $\begin{array}{c} E \\ \downarrow \pi \\ Y \subset X \end{array} \Rightarrow \begin{array}{c} \pi^{-1}(Y) \\ \downarrow \\ Y \end{array}$ is a covering space.

(5.1) **Principle of Monodromy.** Let X be a simply-connected space, and let E be a set together with a set map $E \xrightarrow{\eta} X$. Assume we are given

(1) An open connected set $D \subset X \times X$ containing the diagonal copy of X

(2) $\forall\, (p,q) \in D$, a bijection $\varphi_{qp} : \underset{\overset{\shortparallel}{E_p}}{\eta^{-1}(p)} \longrightarrow \underset{\overset{\shortparallel}{E_q}}{\eta^{-1}(q)}$ s.t.

$$\varphi_{pp} = \mathrm{Id}_{E_p} \quad \text{and} \quad \varphi_{rq} \circ \varphi_{qp} = \varphi_{rp} \quad \forall\, (p,q), (q,r), (p,r) \in D.$$

Then given any $p_0 \in X$ and $e_0 \in E_{p_0}$, $\exists!\ \psi : X \to E$ s.t. $\psi(p_0) = e_0$ and $\psi(q) = \varphi_{qp}(\psi(p))\ \forall\, (p,q) \in D$.

Proof. Introduce a topology on E $\left(= \bigcup_{p \in X} \{p\} \times E_p \text{ as a set}\right)$:

$V \subset E$ is open if for every $(p, e_p) \in V$, $\exists$ an open neighbourhood $\mathcal{U} \subset X$ of p s.t. $\mathcal{U} \times \mathcal{U} \subset D$ and $\forall\, q \in \mathcal{U}, (q, \varphi_{qp}(e_p)) \in V$.

Claim 1: $\begin{array}{c} E \\ \downarrow \eta \\ X \end{array}$ is a covering space.

Assuming this, let $E_0 \subset E$ be the connected component containing (p_0, e_0). $\begin{array}{c} E_0 \\ \downarrow \eta_0 = \eta|_{E_0} \\ X \end{array}$ is a connected covering of X and hence η_0 is homeomorphism. Let $\psi : X \to E_0$ be the inverse of η_0. Clearly, $\psi(p_0) = e_0$

~~Let $X' = \{ q \in X : \psi(q) = \varphi_{qp}(\psi(p))\ \forall\, (p,q) \in D \}$ ($p_0 \in X'$) ~~since~~ We will show that X' is both open and closed in X and hence by connectedness $X' = X$. One uses the exact same argument to prove the uniqueness of ψ.~~

Let $D' \subset D$ consist of pairs (p,q) s.t. $\varphi_{qp}(\psi(p)) = \psi(q)$.

Claim 2. D' is both open and closed in D. ($D' \neq \phi$ since $(p,p) \in D'\ \forall p \in X$)

Since D is connected, $D' = D$. Thus $\varphi_{qp}(\psi(p)) = \psi(q)\ \ \forall (p,q) \in D$ as required.

Uniqueness of ψ. If ψ_1 and ψ_2 are two maps s.t. $\psi_1(p_0) = \psi_2(p_0)$,

$\varphi_{qp}(\psi_\ell(p)) = \psi_\ell(q)$

$(\ell = 1,2)\quad (p,q) \in D$

then the set

$$A := \{ p \in X : \psi_1(p) = \psi_2(p) \}$$

is non-empty, open and closed; hence $A = X$ by connectedness of X.

<u>Proof of Claim 1</u>: For $\mathcal{U} \subset X$ s.t. $\mathcal{U} \times \mathcal{U} \subset D$, and $p \in \mathcal{U}$, $e_p \in E_p$

define $\tilde{\mathcal{U}}(p, \mathcal{U}, e_p) := \{ (q, \varphi_{qp}(e_p)) : q \in \mathcal{U} \}$

Note: every point $p \in X$ has a connected neighbourhood $\mathcal{U}$ s.t. $\mathcal{U} \times \mathcal{U} \subset D$.

Since $(p,p) \in D$ and D is open, $\exists$ an open subset $\underset{\underset{(p,p)}{\Downarrow}}{\mathcal{U}_1 \times \mathcal{U}_2} \subset D$

Take $\mathcal{U} \subset \mathcal{U}_1 \cap \mathcal{U}_2$ to be a connected nhd. of p in X.

Easy checks: (a) $\tilde{\mathcal{U}}(p, \mathcal{U}, e_p) \subset E$ is open

(b) $\eta : E \to X$ is a continuous open map.

(c) E is Hausdorff and locally connected (in fact η sets up a homeo. b/w $\tilde{\mathcal{U}}(p, \mathcal{U}, e_p)$ and $\mathcal{U}$.

$\Rightarrow$ E is a covering space, since every $p \in X$ has a connected nhd $\mathcal{U}$ s.t. $\mathcal{U} \times \mathcal{U} \subset D$ and $\eta^{-1}(\mathcal{U}) = \displaystyle\bigsqcup_{e_p \in E_p} \tilde{\mathcal{U}}(p, \mathcal{U}, e_p)$.

<u>Proof of Claim 2.</u> For any point $(p,q) \in D$ we will construct an open set $V \subset D$ containing (p,q) s.t. either $V \cap D' = \phi$ or $V \subset D'$.

We can find connected nhds $\mathcal{U}_1$ and $\mathcal{U}_2$ of p and q in X s.t.

$$\mathcal{U}_1 \times \mathcal{U}_1 , \quad \mathcal{U}_1 \times \mathcal{U}_2 , \quad \mathcal{U}_2 \times \mathcal{U}_2 \subset D$$

The open set we want is $\mathcal{U}_1 \times \mathcal{U}_2$. If $(\mathcal{U}_1 \times \mathcal{U}_2) \cap D' \neq \phi$, say $(p', q') \in (\mathcal{U}_1 \times \mathcal{U}_2) \cap D'$, then for any $(p,q) \in \mathcal{U}_1 \times \mathcal{U}_2$

$\tilde{\mathcal{U}}(p, \mathcal{U}_1, \psi(p))$ is connected, hence in E_0 $\Rightarrow \psi(p') = \varphi_{p'p}(\psi(p))$

$\tilde{\mathcal{U}}(q, \mathcal{U}_2, \psi(q))$ " " " " $\Rightarrow \psi(q') = \varphi_{q'q}(\psi(q))$

and $(p', q') \in D' \Rightarrow \psi(q') = \varphi_{q'p'}(\psi(p'))$

So, we get

$$\begin{aligned} \varphi_{q'q}(\psi(q)) &= \psi(q') = \varphi_{q'p'}(\psi(p')) \\ &= \varphi_{q'p'}(\varphi_{p'p}(\psi(p))) = \varphi_{q'p}(\psi(p)) \\ &= \varphi_{q'q}(\varphi_{qp}(\psi(p))) \end{aligned}$$

Since $\varphi_{q'q}$ is injective we get $\psi(q) = \varphi_{qp}(\psi(p)) \Rightarrow (p,q) \in D'$.
□

(5.2) Corollary of the principle of monodromy

Let G be a simply connected Lie group, $e \in V \subset G$ a connected nhd. of identity G' another Lie group and $\varphi : V \to G'$ a smooth map s.t. $\varphi(e) = e'$ and $a, b, a.b \in V \Rightarrow \varphi(ab) = \varphi(a)\varphi(b)$. Then φ extends to a unique hom $G \to G'$.

Proof. In the set up of (5.1) take $X = G$. $E = G \times G'$ so that $E_\sigma = G'$ $\forall \sigma \in G$. Define $D = \{(\sigma,\tau) \in G \times G : \tau\sigma^{-1} \in V\} \underset{\substack{\text{open}\\\text{connected}}}{\subset} G \times G$

$$\forall (\sigma,\tau) \in D \qquad \begin{array}{ccc} E_\sigma = G' & \longrightarrow & E_\tau = G' \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \alpha & \longmapsto & \varphi(\tau\sigma^{-1})\alpha \end{array}$$

One can easily verify the hypotheses of (5.1) to conclude that $\exists!$ extension of φ to $G \to G'$ (s.t. $\varphi(e) = e'$). Since V generates G, we can show that $\varphi : G \longrightarrow G'$ is a group hom. As it is smooth near e, it must be smooth everywhere, i.e. a hom of Lie groups $\square$

(5.3) <u>Lifting property</u>: Let $\begin{array}{c} E \\ f\downarrow \\ X \end{array}$ be a covering map and let Y be a simply-connected space, $g : Y \longrightarrow X$ a continuous map. Then $\exists$ $h : Y \to E$ s.t. $f \circ h = g$. If we fix $y_0 \in Y$ and $e_0 \in E$ s.t. $g(y_0) = f(e_0)$, then there is a unique such h s.t. $h(y_0) = e_0$

Proof is entirely analogeous to that of the principle of monodromy. Namely construct $Z = \{(y,e) \in Y \times E : \cancel{h}\ g(y) = f(e)\} \subset Y \times E$

$$\begin{array}{c} Z \\ \downarrow pr_1 \\ Y \end{array}$$

and prove that $\begin{array}{c} Z \\ pr_1 \downarrow \\ Y \end{array}$ is a covering space. Let $Z_0 \subset Z$ be the connected component containing (y_0, e_0). Then $\begin{array}{c} Z_0 \\ \downarrow pr_1 \\ Y \end{array}$ is a homeo

$$\begin{tikzcd} & Z_0 \subset Y \times E \arrow[d, "pr_2"] \arrow[dl, "pr_1"] \\ Y \arrow[ur, bend left] \arrow[r, dashed, "h"] & E \end{tikzcd}$$

. The uniqueness of h also

follows an argument similar to the one in (5.1). Let $h_1, h_2 : Y \to E$ be two lifts s.t. $h_1(y_0) = h_2(y_0)$ for some $y_0 \in Y$. The set of all points $A = \{y \in Y : h_1(y) = h_2(y)\}$ is non-empty, open and closed $\Rightarrow A = Y$ by connectedness of Y. $\square$

(5.4) Topological properties of Lie groups.

(A.) Let G be a connected Lie group and $\begin{array}{c}\tilde{G}\\ \downarrow \pi\\ G\end{array}$ universal covering

Then $\tilde{G}$ has a structure of group, making π a hom of Lie groups.

(B.) $\begin{array}{c}\tilde{G}\\ \eta \downarrow\\ G\end{array}$ universal covering group of G $\Longrightarrow$ $\pi_1(G) = \mathrm{Ker}(\eta)$ is a discrete normal subgroup of $\tilde{G}$.

[HW2 will contain "a discrete normal subgroup of a Lie gp. is contained in its center"]

(C.) G : connected Lie group. $H \subset G$ (connected) closed subgroup

If G/H is simply-connected then we have a surjective map $\pi_1(H) \twoheadrightarrow \pi_1(G)$.

(as we will see in the proof G/H being simply-connected implies H is connected).

(5.5) Proofs of assertions (A) (B) and (C) of (5.4).

(A.) Pick $\tilde{e} \in \pi^{-1}(e)$. To define multiplication and inverse on $\tilde{G}$ (with $\tilde{e} \in \tilde{G}$ being the identity element, we use the lifting property.

$$\tilde{G} \times \tilde{G} \longrightarrow G \times G \xrightarrow[\text{Mult. in } G]{} G$$

As $\tilde{G} \times \tilde{G}$ is also simply-connected we get a lift

$$\begin{tikzcd} & & \tilde{G} \arrow[d] \\ \tilde{G} \times \tilde{G} \arrow[urr, dashed, "\mu"] \arrow[r] & G \times G \arrow[r] & G \end{tikzcd}$$

uniquely determined by the condition that $\mu(\tilde{e}, \tilde{e}) = \tilde{e}$.

Similarly for the inverse.

(B.) $\begin{tikzcd} \tilde{G} \arrow[d, "\eta"'] \\ G \end{tikzcd}$ universal covering

$$\pi_1(G) = \left\{ f : \tilde{G} \to \tilde{G} \text{ s.t. } \eta \circ f = \eta \right\} \longleftrightarrow \operatorname{Ker}(\eta)$$

$$f \longmapsto f(e) \in \eta^{-1}(e)$$

$$\text{left mult. by } \sigma \longleftarrow\!\shortmid\ \sigma \in \eta^{-1}(e)$$

[remark in (B.) of (5.4) $\Rightarrow$ $\pi_1(G)$ is abelian]

(C.) $H \subset G$ (H: closed subgroup; G: connected Lie gp.). Let $\tilde{G}$ and $\tilde{H}$ be universal covering groups of G and H respectively

$$\begin{tikzcd} & \eta^{-1}(H) \arrow[d, "\bar{\eta}"] \arrow[r, phantom, "\subset"] & \tilde{G} \arrow[d, "\eta"] \\ \tilde{H} \arrow[ur, dashed, "\xi"] \arrow[r, "\nu"'] & H \arrow[r, phantom, "\subset"] & G \end{tikzcd}$$

Lifting prop. (unique $\tilde{e} \mapsto e$)

Claim: G/H simply-connected $\Rightarrow$ H is connected

Apply the claim to $\tilde{G}/\eta^{-1}(H) \simeq G/H$ is simply-connected

$\Rightarrow$ $\eta^{-1}(H)$ is connected (covering of H by Fact (3) of (5.0)).

$\Rightarrow$ ξ is surjective and we get a map $\pi_1(H) \longrightarrow \pi_1(G)$ as:

Let $a \in \mathrm{Ker}(\nu) = \pi_1(H)$. Then $\xi(a) \in \tilde{G}$ belongs to $\mathrm{Ker}(\eta) = \pi_1(G)$.

~~Conversely~~ If $b \in \mathrm{Ker}(\eta) = \eta^{-1}(e)$ then $b \in \eta^{-1}(H)$ s.t. $\eta(b) = e$.

By surjectivity of ξ, we can find $a \in \tilde{H}$ s.t. $\xi(a) = b$. Then

$\nu(a) = \cancel{\eta(\xi(a))} = \eta(\xi(a)) = \eta(b) = e \Rightarrow a \in \mathrm{Ker}(\nu) = \pi_1(H)$.

Hence $\pi_1(H) \twoheadrightarrow \pi_1(G)$.

<u>Idea of the proof of the claim</u>: Let $H_0 \subset H$ be the connected component of $e \in H$. Then $H_0 \subset H$ is a normal subgroup and we can prove that

$$\begin{array}{c} G/H_0 \\ \downarrow \\ G/H \end{array}$$

is a connected covering map. G/H being simply-conn.

this map is bijective $\Rightarrow$ $H = H_0$ is connected. $\square$