Lecture 6

(6.0) Recall: we proved two main results about fundamental group of a connected Lie group G.

- $\begin{array}{c}\tilde{G}\\ \downarrow \eta \\ G\end{array}$ $\quad \tilde{G}$: simply-connected and η a covering map

  $\pi_1(G) = \operatorname{Ker}\eta$ is a discrete normal subgroup of $\tilde{G}$

Fact (HW2) Every discrete normal subgroup of a connected Lie group is contained in its center, hence abelian.

- $H \subset G$ closed subgroup $\quad G/H$ simply-connected space

  $\Longrightarrow$ H is connected and $\pi_1(H) \twoheadrightarrow \pi_1(G)$ (surjective)

Lemma: H and G/H connected $\Rightarrow$ G is connected
(H closed subgp of G)

Proof. Assume the contrary, $G = \mathcal{U} \sqcup V$ disjoint union of two open sets $\mathcal{U}, V \subset G$. Then $G \xrightarrow{\pi} G/H$ is open map (see (6.5)) and G/H connected $\Rightarrow$ $\pi(\mathcal{U}) \cap \pi(V) \neq \phi$.

So $\exists\, g \in G$ s.t. the coset $g.H \in \pi(\mathcal{U}) \cap \pi(V)$

i.e. $\exists\, h_1, h_2 \in H$ s.t. $gh_1 \in \mathcal{U}$, $gh_2 \in V$ and hence

$g^{-1}\mathcal{U} \cap H$, $g^{-1}V \cap H$ are both non-empty and cover H (& disjoint)

contradicts connectedness of H $\square$

(6.1) Example of $SO_n(\mathbb{R})$.

Recall: $A \in SO_n(\mathbb{R}) \iff A^T A = \mathrm{Id}$ and $\det A = 1$

i.e. columns of A have square length 1 and distinct columns are orthogonal

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

$(v_i, v_j) = \delta_{ij}$ and "orientation is preserved":

($\text{i.e. } v_1 \wedge \dots \wedge v_n = e_1 \wedge \dots \wedge e_n$)

e.g. $SO_2(\mathbb{R}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a^2 + b^2 = 1$

so, $SO_2(\mathbb{R}) \underset{\text{homeo}}{\simeq} S^1 \leftarrow$ compact, connected and $\pi_1(SO_2(\mathbb{R})) = \mathbb{Z}$.

$SO_n(\mathbb{R})$ is compact since it is a closed subset of $\underbrace{S^{n-1} \times \dots \times S^{n-1}}_{n\text{-terms}}$.

Consider the map

$$SO_n(\mathbb{R}) \longrightarrow S^{n-1}$$
$$A \longmapsto A \cdot e_1 \quad (1^{st} \text{ col. of } A)$$

This map is surjective and $SO_{n-1}(\mathbb{R}) = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix} \subset SO_n(\mathbb{R})$ is the subgroup mapping to $(1, 0 \dots 0) \in S^{n-1}$

$\Rightarrow SO_n(\mathbb{R}) / SO_{n-1}(\mathbb{R}) \simeq S^{n-1}$ by induction (& using Lemma (6.0)) $SO_n(\mathbb{R})$ is connected.

(6.2) Example of SU_n.

U_n = set of $n \times n$ matrices w/ entries from $\mathbb{C}$, say $X \in M_{n\times n}(\mathbb{C})$

s.t. $\overline{X}^t = X^{-1}$

SU_n : moreover $\det X = 1$.

e.g. $U_1 = \{ z \in \mathbb{C} : \bar{z}.z = 1 \} \simeq S^1$

$\cup$

$SU_1 = \{1\}$

$$SU_2 = \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} : \begin{array}{c} \alpha, \beta \in \mathbb{C} \\ |\alpha|^2 + |\beta|^2 = 1 \end{array} \right\} \simeq S^3$$

Note : each column of U_n consists of n complex numbers $\alpha_1, \dots, \alpha_n$ s.t. $\sum_{j=1}^{n} |\alpha_j|^2 = 1 \quad \in S^{2n-1}$. Hence U_n is compact. SU_n is closed in U_n hence compact as well.

[HW2] $$SU_n / SU_{n-1} \simeq S^{2n-1}$$

Hence $\{SU_n\}_{n \geq 2}$ are all connected, simply-connected (and compact).

These are 3 topological properties we desire the most !!

(6.3) Exponential Map.

$\mathfrak{g} = \mathrm{Lie}(G)$ $\quad$ G = a connected Lie group.

Let $X \in \mathfrak{g}$ and consider the hom. of Lie algebras
$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathfrak{g} \\ \Psi & & \Psi \\ t & \longmapsto & tX \end{array}$$

$\mathbb{R}$ = Lie algebra of the Lie group $(\mathbb{R}, +)$ which is simply-connected

By Lie's Thm $\exists!$ hom of Lie groups
$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & G \\ \Psi & & \Psi \\ t & \longrightarrow & \exp(tX) \end{array}$$
called $\underline{\exp(tX)}$

At $t=1$, we get an element $\exp(X) \in G$.

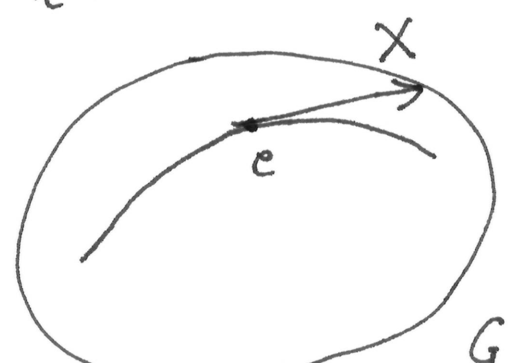
inducing
$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathfrak{g} \\ \Psi & & \Psi \\ t & \longmapsto & tX \end{array}.$$

- $\exp(0 \cdot X) = e$ $\qquad$ $\exp((t_1 + t_2)X) = \exp(t_1 X) . \exp(t_2 X)$

since this map $\mathbb{R} \to G$ is a hom. of Lie groups [One parameter subgp.]

Geometrically $\mathbb{R} \longrightarrow G$ defines a path through $e \in G$ s.t.

$$\frac{d}{dt} \exp(tX) \Big|_{t=0} = X \in \mathfrak{g} = T_e G.$$

$\mathbb{R} \longmapsto$ 

$\underline{\text{These conditions}}$ define $\exp(tX)$ uniquely (by Lie's Thm.)

- it is 1-parameter subgp. of G
- $t = 0 \rightsquigarrow e \in G$
- $\frac{d}{dt} \exp(tX) \Big|_{t=0} = X$

Taking $t = 1$ gives a map

$$\begin{tikzcd} X \arrow[d, mapsto] & \in & \mathfrak{g} \arrow[d, "\exp"] \\ \exp(X) & \in & G \end{tikzcd}$$

Prop. (1) $\exp : \mathfrak{g} \longrightarrow G$ is smooth and $T_0 \exp : \underset{\mathfrak{g}}{\underset{\|}{T_0 \mathfrak{g}}} \longrightarrow \underset{\mathfrak{g}}{\underset{\|}{T_e G}}$ is identity. Hence exp is a local diffeo.

(2) Naturality : $G \xrightarrow{\varphi} G'$ hom. of Lie gps $\xi = \mathrm{Lie}(\varphi) : \mathfrak{g} \to \mathfrak{g}'$ hom. of Lie algebras. Then $\varphi(\exp(X)) = \exp(\xi(X))$ $\forall X \in \mathfrak{g}$.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\xi"] \arrow[d, "\exp"'] & \mathfrak{g}' \arrow[d, "\exp"] \\ G \arrow[r, "\varphi"] & G' \end{tikzcd}$$

e.g. for matrix groups (X is $n \times n$ matrix w/ real or complex entries)

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} = e^X .$$

Reason : $t \in \mathbb{R} \longrightarrow e^{tX}$ is group hom satisfying $0 \mapsto 1$ and $\left.\frac{d}{dt} e^{tX}\right|_{t=0} = X$. Claim follows from uniqueness.

(6.4) Adjoint repn. G = a Lie group $\mathfrak{g} = \mathrm{Lie}(G)$

$\forall \sigma \in G$ let $\mathrm{Conj}(\sigma) : G \to G$, $\alpha \mapsto \sigma \alpha \sigma^{-1}$ is a hom. of Lie groups $\rightsquigarrow$ $\mathrm{Ad}(\sigma) := \mathrm{Lie}(\mathrm{Conj}(\sigma)) : \mathfrak{g} \to \mathfrak{g}$ is a hom. of Lie alg.

conjugation by σ

Easy check: $\sigma \in G \longmapsto \underbrace{\mathrm{Ad}(\sigma) : \mathfrak{g} \longrightarrow \mathfrak{g}}_{\text{Lie alg. hom}}$

is a group hom.

i.e. $\mathrm{Ad}(e) = \mathrm{Id}_{\mathfrak{g}}$ $\qquad \mathrm{Ad}(\sigma\sigma') = \mathrm{Ad}(\sigma) \circ \mathrm{Ad}(\sigma')$

$\mathrm{Ad}(\sigma^{-1}) = \mathrm{Ad}(\sigma)^{-1}$

hence $\mathrm{Ad}(\sigma)$ is invertible linear map.

If $X \in \mathfrak{g}$, we can define a linear map

$$\mathrm{ad}(X) : \mathfrak{g} \longrightarrow \mathfrak{g} \qquad \in \mathrm{End}(\mathfrak{g})$$
$$Y \longmapsto [X,Y]$$

(linear endomorphisms of $\mathfrak{g}$)

Claim: $\mathrm{ad}([X,Y]) = \mathrm{ad}X\, \mathrm{ad}Y - \mathrm{ad}Y\, \mathrm{ad}X$

i.e. $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is a hom. of Lie algebras

Pf.
$$\begin{aligned} \mathrm{ad}([X,Y])(Z) &= [[X,Y],Z] \\ &= [X[Y,Z]] - [Y,[X,Z]] \quad \text{(Jacobi identity)} \\ &= \mathrm{ad}X(\mathrm{ad}Y(Z)) - \mathrm{ad}Y(\mathrm{ad}X(Z)) \end{aligned}$$

Jacobi identity is also equivalent to: $\mathrm{ad}(X)$ is a derivation

i.e. Leibniz rule holds

$$(\mathrm{ad}X)([Y,Z]) = [\mathrm{ad}X(Y), Z] + [Y, \mathrm{ad}X(Z)]$$

(6.5) Some remarks about quotient G/H.

Let G be a Lie group and $H \subset G$ a closed subgroup.

$X = G/H$ = the set of left H-cosets $\{\sigma H : \sigma \in G\}$

A subset $V \subset X$ is <u>open</u> iff $\pi^{-1}(V) \subset G$ is open.

$G \xrightarrow{\pi} G/H$ is an open map, i.e. $\forall\, \mathcal{U} \subset G$ open, $\pi(\mathcal{U}) \subset G/H$ is open. This is because $\pi^{-1}(\pi(\mathcal{U})) = \bigcup_{h \in H} \mathcal{U}\cdot h$ is a union of open sets, hence open.

Remark. We need H to be closed, for X to be Hausdorff. For the case of Lie groups the proof of this assertion is almost trivial, since G is <u>normal</u>, i.e. any two closed subsets $K_1, K_2 \subset G$ s.t. $K_1 \cap K_2 = \phi$, admit disjoint neighbourhoods: i.e. $\mathcal{U}_1, \mathcal{U}_2 \underset{\text{open}}{\subset} G$ s.t. $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$, $K_\ell \subset \mathcal{U}_\ell$ $(\ell = 1,2)$.

So, if $x_1 \neq x_2 \in X = G/H$, $\pi^{-1}(x_1) = \sigma_1 H$ & $\pi^{-1}(x_2) = \sigma_2 H$ are closed, disjoint subsets of G. Pick $\mathcal{U}_\ell \supset \sigma_\ell H$ $(\ell = 1,2)$ open subsets of G s.t. $\mathcal{U}_1 \cap \mathcal{U}_2 = \phi$. Then $V_\ell = \pi(\mathcal{U}_\ell)$ are disjoint open sets in X s.t. $x_\ell \in V_\ell$ $(\ell = 1,2)$.

(6.6) Proof of Prop (6.3) on page 5 above. (part (2) is obvious)

(1) $\mathfrak{g} \xrightarrow{\exp} G$ is smooth

Recall we have $\begin{array}{ccc} \mathbb{R} \times \mathfrak{g} & \longrightarrow & G \\ (t, X) & \longmapsto & \exp(tX) \end{array}$ uniquely determined

by: for $X \in \mathfrak{g}$, $\begin{array}{c} \mathbb{R} \to G \\ t \mapsto \exp(tX) \end{array}$ is smooth gp. hom. s.t. $\left.\dfrac{d}{dt}\exp(tX)\right|_{t=0} = X$.

Choose a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ and a coord. nhd. of $e \in G$ $(u_1, \dots, u_n) : \mathcal{U} \longrightarrow \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^n$. We get n^2 $\mathbb{R}$-valued fns. on $\mathrm{Cube}_r(\underline{0})$, defined by $X_i(u_j)(\sigma) =: F_{ij}(u_1(\sigma), \dots, u_n(\sigma))$. Then for a given $X = \sum_{i=1}^n a_i X_i \in \mathfrak{g}$, the functions $\varphi_i(t, \underline{a}) := u_i(\exp(tX))$ (defined on $(t, \underline{a}) \in \mathbb{R}^{n+1}$ s.t. $\exp(t \sum a_j X_j) \in \mathcal{U}$) are solutions of

$$\left.\begin{aligned} \frac{d}{dt}\varphi_j(t, \underline{a}) &= \sum_{j=1}^n a_i F_{ij}(\varphi_1(t; \underline{a}), \dots, \varphi_n(t; \underline{a})) \\ \varphi_j(0, \underline{a}) &= 0 \end{aligned}\right\} \forall\, 1 \le j \le n$$

Again we invoke existence and uniqueness of solutions of ODE's to prove that each φ_j is smooth in some cubic neighbourhood $(t, \underline{a}) \in \mathrm{Cube}_s(\underline{0}) \subset \mathbb{R}^{n+1}$.

Now $\varphi_j(t, \underline{a}) = \varphi_j(1, t\underline{a})$ can be computed as long as the last n coordinates are within $\mathrm{Cube}_{s^2}(\underline{0}) \subset \mathbb{R}^n$. This proves exp is smooth near $0 \in \mathfrak{g}$. For any bounded open set V in $\mathfrak{g}$ (open nhd. of some $Y \in \mathfrak{g}$) we can rescale $M^{-1}V \subset \mathrm{Cube}_{s^2}(\underline{0})$ where exp has been proved to be smooth and use $\exp(Y) = \exp(M^{-1}Y)^M$ to prove that exp is smooth everywhere.

$\square$