Lecture 7

(7.0) Recall : G = connected Lie group $\mathfrak{g} = \mathrm{Lie}(G)$. Last time we introduced

- $\exp : \mathfrak{g} \to G$ smooth map, diffeo near $0 \in \mathfrak{g}$. This was defined using $\mathbb{R} \times \mathfrak{g} \to G$, $(t, X) \mapsto \exp(tX)$ where for fixed $X \in \mathfrak{g}$, $\mathbb{R} \to G$, $t \mapsto \exp(tX)$ is <u>the unique 1-parameter subgroup</u> s.t. $\left.\frac{d}{dt}\exp(tX)\right|_{t=0} = X$.

- $\mathfrak{gl}(\mathfrak{g})$ or $\mathrm{End}(\mathfrak{g})$ = vector space of all $\mathbb{R}$-linear maps $\mathfrak{g} \to \mathfrak{g}$ (Lie algebra under $[A,B] = AB - BA$).

$$\cup$$

$$\mathrm{Der}(\mathfrak{g}) = \{ f : \mathfrak{g} \to \mathfrak{g} \text{ s.t. } f([A,B]) = [f(A), B] + [A, f(B)] \}$$

(derivations : Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$)

$\mathrm{Aut}(\mathfrak{g})$ or $GL(\mathfrak{g})$ = group of all invertible linear maps $\mathfrak{g} \to \mathfrak{g}$

$$\cup$$

$$\mathrm{Aut}_{L.A.}(\mathfrak{g}) = \{ \varphi \in GL(\mathfrak{g}) : \varphi([A,B]) = [\varphi(A), \varphi(B)]$$

We defined a hom. of Lie groups $G \xrightarrow{Ad} \mathrm{Aut}_{L.A.}(\mathfrak{g})$

(Ad is smooth by HW2, at $e \in G$ and hence everywhere ← using left translations).

Lie algebra of $\mathrm{Aut}_{LA}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ (also by HW2)

and $\mathrm{Lie}(Ad) = ad$

Recall $Ad(\sigma) : \mathfrak{g} \to \mathfrak{g}$ is the derivative at e of conjugation map $\alpha \mapsto \sigma \alpha \sigma^{-1}$ from $G \to G$. $ad(X) : \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X,Y]$.

(7.1) One application of exp (G : connected Lie group $\mathfrak{g} = \mathrm{Lie}(G)$)

Let $\{X_1, \dots, X_n\}$ be a basis of $\mathfrak{g}$. Since exp is a local diffeo. we can find a coord. neighbourhood $\mathcal{U}$ of $e \in G$ and inverse to exp on $\mathcal{U}$, In other words :

- $\exists\ r > 0$ s.t. $\exp\left(\sum_{i=1}^{n} a_i X_i\right) \in \mathcal{U} \iff |a_i| < r \quad \forall\ 1 \le i \le n.$
- $u_i\left(\exp \sum_{j=1}^{n} a_j X_j\right) = a_i$ are the coordinates in $\mathcal{U}$

$$(u_1, \dots, u_n) : \mathcal{U} \longrightarrow \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^n \underset{\uparrow}{\simeq} \mathfrak{g}$$

by fixing basis $\{X_1, \dots X_n\}$

We refer to $(u_1, \dots, u_n)$ as <u>coordinates adapted to basis $\{X_1, \dots, X_n\}$</u>.

Prop. Let $\varphi : G \to G'$ be a continuous group hom of two Lie groups. Then φ is smooth.

Proof Step 1 : Prove the proposition for $G = \mathbb{R}$, $G' = H$. i.e. given a continuous gp. hom. $\eta : \mathbb{R} \longrightarrow H$, we claim that η is smooth.

In fact we will prove that we can find $X \in \mathfrak{h}$ s.t. $\eta(t) = \exp(tX)$.

Choose a coord. system near $e \in H$ adapted to a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{h}$ ($n = \dim \mathfrak{h}$). , say

$$\begin{array}{ccccl} (u_1, \dots, u_n) : \mathcal{U} & \xrightarrow{\sim} & \mathrm{Cube}_r(\underline{0}) & \subset \mathbb{R}^n \simeq \mathfrak{h}. \\ \cup & & \cup & \\ \mathcal{U}' & \xrightarrow{\sim} & \mathrm{Cube}_{r'}(\underline{0}) & \text{(choose some } 0 < r' < r) \end{array}$$

Let $t_0 \in \mathbb{R}$, $t_0 \neq 0$ be such that $\eta(t_0) \in \mathcal{U}'$ and fix $X_0 \in \mathfrak{h}$ s.t.

$$\eta(t_0) = \exp(X_0) \quad \left(\text{here } X_0 = \sum_{j=1}^{n} u_j(\eta(t_0)) \cdot X_j \text{ by defn of } \underline{u}\right)$$

We claim that $\eta(t) = \exp\left(\frac{t}{t_0} X_0\right)$. We write this as

$\eta(r \cdot t_0) = \exp(r X_0)$ $\forall\, r \in \mathbb{R}$. The idea is to prove this for rational

$r \in \mathbb{Q}$ of size < 1.

eg. if $r = \frac{1}{N}$ then let $\sigma = \eta\left(\frac{t_0}{N}\right)$. As $\sigma^N \in \mathcal{U}$, we know that

$u_i(\sigma^k) = k\, u_i(\sigma)$ are all in $\mathcal{U}$ $(0 \leq k \leq N)$ and hence

$u_i(\sigma^N) = N\, u_i(\sigma) \Rightarrow u_i(\sigma) = \frac{1}{N} u_i(\sigma^N)$ and thus $u_i(\sigma^k) = \frac{k}{N} u_i(\sigma^N)$

i.e. $u_i\left(\eta\left(\frac{k}{N} t_0\right)\right) = \frac{k}{N} u_i\left(\eta(t_0)\right)$ $(0 \leq k \leq N)$. The same is true

for $\exp\left(\frac{k}{N} X_0\right)$. Thus $\eta(r \cdot t_0) = \exp(r \cdot X_0)$ for rat'l r of size <1

By continuity $\eta(r \cdot t_0) = \exp(r \cdot X_0)$ $\forall\, r \in \mathbb{R}$, $|r| \leq 1$. Since both

sides are gp. hom $\eta(t \cdot t_0) = \exp(t \cdot X_0)$ $\forall\, t \in \mathbb{R}$ and we are done.

Step 2. General case. $\varphi : G \to G'$ is a continuous gp. hom.

Let $\{X_1, \ldots, X_m\}$ be a basis of $\mathfrak{g}$ $(m = \dim \mathfrak{g})$.

We get $\begin{array}{ccc} \mathbb{R} & \longrightarrow & G' \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ t & \longmapsto & \varphi(\exp(t X_i)) \end{array}$. By previous part $\exists\, Y_i \in \mathfrak{g}'$ s.t.

$\forall\, 1 \leq i \leq m$:

$$\varphi(\exp(t X_i)) = \exp(t Y_i) \qquad (1 \leq i \leq m)$$

and $\varphi\left(\exp(t_1 X_1) \cdots \exp(t_n X_n)\right) = \exp(t_1 Y_1) \cdots \exp(t_n Y_n)$ since φ

is a group hom

Let $(u_1, \ldots, u_m) : \mathcal{U} \longrightarrow \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^m \simeq \mathfrak{g}$ be coord. adapted to $\{X_1, \ldots, X_n\}$

For a smaller open nbhd of $\underline{0} \in \mathrm{Cube}_r(\underline{0})$, the map

$$(t_1, \dots, t_m) \longmapsto \exp(t_1 X_1) \dots \exp(t_n X_m) \in \mathcal{U}$$

and $f_j(t_1, \dots, t_m) = u_j\left(\exp(t_1 X_1) \cdots \exp(t_n X_m)\right)$ are smooth with

$\left.\dfrac{\partial f_i}{\partial t_j}\right|_{\underline{0}} = \delta_{ij}$ (by defn. of $u_1, \dots, u_m$). This allows us to introduce

$t_1, \dots, t_m$ as local coordinates near $e \in G$, and in these coordinates the hom φ has the form

$$(t_1, \dots, t_m) \longmapsto \exp(t_1 Y_1) \dots \exp(t_m Y_m)$$ which is

smooth. Since φ is smooth near $e \in G$ and a gp. hom. it is smooth everywhere $\square$

(7.2) <u>Cor.</u> If G is a topological group then there is at most one smooth structure on G making it a Lie group.

(7.3) One more example. $Sp(n)$. ↓ didn't cover

Introduce a non-degenerate symplectic form (skew-symmetric) on $\mathbb{C}^{2n}$ in term of a basis $\{\varepsilon_1, \dots, \varepsilon_n; \eta_1, \dots, \eta_n\}$:

$$(\varepsilon_i, \varepsilon_j) = 0 = (\eta_i, \eta_j) \qquad (\varepsilon_i, \eta_j) = \delta_{ij} = -(\eta_j, \varepsilon_i)$$

$$Sp(n; \mathbb{C}) := \left\{ A \in M_{2n \times 2n}(\mathbb{C}) : \begin{array}{c}(Av, Aw) = (v, w) \\ \forall\, v, w \in \mathbb{C}^{2n}\end{array} \right\}$$

$Sp(n) := Sp(n;\mathbb{C}) \cap U(2n)$ closed subgroup of $U(2n)$, hence compact.

Lemma $\{Sp(n)\}_{n\geq 1}$ are simply connected.

Proof: $Sp(1) = \begin{bmatrix} \alpha & -t\bar{\beta} \\ \beta & t\bar{\alpha} \end{bmatrix} = A$ s.t. $A^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

a typical matrix in $U(2)$

here $|\alpha|^2 + |\beta|^2 = 1$, $|t| = 1$

Last cond$^{\underline{n}}$ $\Rightarrow \det A = 1 = t$

Hence $Sp(1) \simeq SU(2) \simeq S^3$ simply-connected.

Claim: $Sp(n)/Sp(n-1) \simeq S^{4n-1}$ $(n \geq 2)$

Proof left as an exercise.

Hence by results of Lecture 5, each $Sp(n)$ is simply-connected $\square$

(7.4) Examples we have seen so far (all compact)

Simply-connected	Connected but not s-c.	Not connected
$SU(n)$	$\subset U(n)$	
	$SO_n(\mathbb{R})$	$\subsetneq O_n(\mathbb{R})$
$Sp(n)$	$\subset U(2n)$	

(We haven't yet computed $\pi_1(SO_n(\mathbb{R}))$)