Lecture 8 - Review of Lie algebras

(8.0) Today we will study (abstract) Lie algebras. The base field for us is $\mathbb{R}$ or $\mathbb{C}$, unless specified.

Recall: a Lie algebra $\mathfrak{g}$ is a vector space together with a bilinear $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ s.t. (1) $[X,Y] = -[Y,X]$

(2) $$[[X,Y],Z] = [X[Y,Z]] - [Y,[X,Z]]$$

Let $\mathfrak{g}' \subset \mathfrak{g}$ be a subspace. We say $\mathfrak{g}'$ is a subalgebra (or ideal resp.) if $\forall X,Y \in \mathfrak{g}'$, $[X,Y] \in \mathfrak{g}'$ (or, $\forall \begin{smallmatrix} X \in \mathfrak{g} \\ Y \in \mathfrak{g}' \end{smallmatrix}$, $[X,Y] \in \mathfrak{g}'$ resp.)

For $X \in \mathfrak{g}$, we defined $ad(X) : Y \longmapsto [X,Y]$ as a linear map (also a derivation) $\mathfrak{g} \to \mathfrak{g}$ and proved that $[ad(X), ad(Y)] = ad([X,Y])$

Definition A representation of $\mathfrak{g}$ is a vector space V and a hom. of Lie algebras $\rho_V : \mathfrak{g} \longrightarrow End(V)$

i.e. $\rho([X,Y]) = [\rho(X), \rho(Y)]$

I will often suppress ρ and write $\mathfrak{g} \circlearrowright V$ ($\mathfrak{g}$ acts on V) and for $X \in \mathfrak{g}$, $v \in V$, $X \cdot v = \rho(X)(v) \in V$.

For this lecture $\mathfrak{g}$ and its representation V will be assumed to be finite-dimensional.

e.g. $V = \mathfrak{g}$ and $\rho = ad$ $\leftarrow$ adjoint repn. of $\mathfrak{g}$.

(8.1) If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are two ideals, then $[\mathfrak{a}, \mathfrak{b}] := \{[X,Y] : X \in \mathfrak{a}, Y \in \mathfrak{b}\}$ is again an ideal.

linear span of

Defn. We say $\mathfrak{g}$ is <u>abelian</u> if $[X,Y] = 0 \ \forall \ X, Y \in \mathfrak{g}$.

<u>Solvable</u> if $\mathcal{D}^N(\mathfrak{g}) = 0$ for $N \gg 0$. Here $\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$, $\mathcal{D}^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ ... $\mathcal{D}^l(\mathfrak{g}) = [\mathcal{D}^{l-1}(\mathfrak{g}), \mathcal{D}^{l-1}(\mathfrak{g})]$ is a descending chain of ideals

$$\mathfrak{g} \supset \mathcal{D}^1(\mathfrak{g}) \supset \dots \supset \mathcal{D}^N(\mathfrak{g}) = 0 \qquad [\text{Solvable}]$$

Note: if N is the smallest such number, i.e. $\mathcal{D}^{N-1}\mathfrak{g} \neq 0$, $\mathcal{D}^N\mathfrak{g} = 0$, then $\mathcal{D}^{N-1}\mathfrak{g}$ is an abelian (non-zero) ideal, in $\mathfrak{g}$.

<u>Nilpotent</u> if $\mathcal{C}^N(\mathfrak{g}) = 0$ for $N \gg 0$. Here $\mathcal{C}^0\mathfrak{g} = \mathfrak{g}$, $\mathcal{C}^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ and $\mathcal{C}^l\mathfrak{g} = [\mathcal{C}^{l-1}\mathfrak{g}, \mathfrak{g}]$ $\left(= [\mathfrak{g}, \mathcal{C}^{l-1}\mathfrak{g}]\right)$ is another descending chain of ideals $\mathfrak{g} \supset \mathcal{C}^1\mathfrak{g} \supset \dots \supset \mathcal{C}^N\mathfrak{g} = 0$. [Nilpotent]

Note: if $\mathfrak{g}$ is nilpotent, then $\forall \ X \in \mathfrak{g}$, $ad(X)^N(Y) \in \mathcal{C}^N\mathfrak{g} = 0$
$\Rightarrow$ $ad(X)$ is a nilpotent operator

e.g. $\mathfrak{g} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \text{ (or } \mathbb{C}) \right\}$. Then $\mathcal{C}^1\mathfrak{g} = \mathcal{D}^1\mathfrak{g}$ $= \left\{ \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} : d \in \mathbb{R} \right\}$

$\mathcal{D}^2\mathfrak{g} = 0 \qquad \mathcal{C}^2\mathfrak{g} = \mathcal{C}^1\mathfrak{g} \ (= \mathcal{C}^3\mathfrak{g} = \dots)$

So $\mathfrak{g}$ is solvable but not nilpotent

(8·2) Lemma. (1) Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal. Then $\mathfrak{g}$ is solvable iff $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are solvable.

(2) Let $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ be two solvable ideals. Then $\mathfrak{a}+\mathfrak{b}$ is solvable

Proof (2) follows from (1) since $\mathfrak{a}$, $\mathfrak{a}+\mathfrak{b}/\mathfrak{a}$ are solvable.

(1): ($\Rightarrow$)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{g} & \xrightarrow{p} & \mathfrak{g}/\mathfrak{a} & \longrightarrow & 0 \\
 & & \cup & & \cup & & \cup & & \\
 & & \mathcal{D}^1(\mathfrak{a}) & & \mathcal{D}^1(\mathfrak{g}) & & \mathcal{D}^1(\mathfrak{g}/\mathfrak{a}) & & \\
 & & \cup & & \cup & & \cup & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

this is equality since p is hom of Lie alg

Note $\mathcal{D}^l(\mathfrak{a}) \subset \mathfrak{a} \cap \mathcal{D}^l(\mathfrak{g})$ and $\mathcal{D}^l(\mathfrak{g}/\mathfrak{a}) \subset p(\mathcal{D}^l(\mathfrak{g}))$ ✓

($\Leftarrow$) if $\mathcal{D}^n(\mathfrak{g}/\mathfrak{a}) = 0$ $\forall\, n \geq N$, then $\mathcal{D}^n(\mathfrak{g}) \subset \mathrm{Ker}(p) = \mathfrak{a}$

so $\mathcal{D}^{N+k}(\mathfrak{g}) \subset \mathcal{D}^k(\mathfrak{a}) = 0$ for $k \gg 0$, and we are done. □

(8·3) Let $\mathrm{rad}(\mathfrak{g})$ = sum of all solvable ideals of $\mathfrak{g}$ (again solvable by Lemma (8·2) above).

We say $\mathfrak{g}$ is <u>semisimple</u> if $\mathrm{rad}(\mathfrak{g}) = 0$. Equivalently if $\mathfrak{g}$ does not contain any non-zero abelian ideals (see Note after defn. of solvable Lie alg. in (8·1) of page 2).

$\mathfrak{g}$ is <u>simple</u> if it has no proper, non-zero ideals. i.e.

$\underset{\text{ideal}}{\mathfrak{a}} \subset \underset{\text{simple}}{\mathfrak{g}} \Rightarrow \mathfrak{a} = 0$ or $\mathfrak{g}$.

[Here it is customary to <u>exclude</u> $\dim \mathfrak{g} = 1$ case. One-dim'l trivial Lie alg is <u>NOT</u> simple]

(8.4) Lie's Theorem for solvable Lie algebras.

as usual.

$\mathfrak{g}$: solvable Lie alg. / $\mathbb{C}$. $\mathfrak{g} \circlearrowright V$ a (f.d.) repn. Then

$\exists \underset{(v \neq 0)}{v \in V},\ \lambda \in \mathfrak{g}^*$, s.t. $X.v = \lambda(X)v \quad \forall X \in \mathfrak{g}$.

dual vector space $= \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$

[We can make the same assertion over $\mathbb{R}$ if eigenvalues of all linear operators $\underset{(X \in \mathfrak{g})}{X \circlearrowright V}$ are real].

Proof. By induction on $\dim \mathfrak{g}$. Base case is trivially true. Now $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$. In fact we can pick an ideal $\mathfrak{h} \subset \mathfrak{g}$ s.t. $\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}]$, $\dim \mathfrak{g}/\mathfrak{h} = 1$. This is because if $\mathfrak{h}$ is any such subspace, then

$$[\mathfrak{g}, \mathfrak{h}] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}.$$

By induction $\exists\ 0 \neq e \in V$, $\lambda \in \mathfrak{h}^*$ st. $H.e = \lambda(H)v \quad \forall H \in \mathfrak{h}$.

Take $X \in \mathfrak{g}$ s.t. $\mathfrak{g} = \langle X, \mathfrak{h} \rangle$. $V_1 = \mathrm{Span\ of}\ \{ \underset{v_0}{\overset{\parallel}{e}}, \underset{v_1}{\overset{\parallel}{Xe}}, \underset{v_2}{\overset{\parallel}{X^2e}}, \dots \}$

lin. span of

Claim $\quad H.v_j = \cancel{\lambda(v_j)}\ \lambda(H).v_j$

Given the claim, pick any eigenvector of X acting on V_1 which will be joint eigenvector for $\{X, H : H \in \mathfrak{h}\} = \mathfrak{g}$.

<u>Proof of the claim</u>: We first show that $H.v_j = \lambda(H)v_j +$ lower terms

in $\mathrm{Span}\{v_0, \dots, v_{j-1}\}$

This is true for $j=0$. After that $(j \geq 1)$

$$H \cdot v_j = HXv_{j-1} = XHv_{j-1} + [HX]v_{j-1}$$

$$= X \cdot (\lambda(H)v_{j-1} + \cdots) + \lambda([HX])v_{j-1} + \cdots \quad (\cdots \in \mathrm{Span}\{v_0 \cdots v_{j-2}\})$$

$$= \lambda(H)v_j + (\text{in the span of } v_0, \ldots, v_{j-1})$$

Thus $\mathrm{Trace}_{V'}(H) = \lambda(H) \cdot \dim V'$. For $[H,X] \in \mathfrak{h}$, we know $\mathrm{Trace}_{V'}([H,X]) = 0$. So $\lambda([H,X]) = 0 \quad \forall H \in \mathfrak{h}$. This will finish the proof of the claim as follows. The claim is true for $j=0$.

Then $(j \geq 1)$

$$H \cdot v_j = HXv_{j-1} = XHv_{j-1} + [HX]v_{j-1}$$

$$= \lambda(H)v_j + \underbrace{\cancel{\lambda([H,X])}}_{0} v_{j-1} = \lambda(H)v_j$$

□

(8.5) Cor. V admits a flag of subspaces stable under $\mathfrak{g}$
[Hypotheses of Thm (8.4)] $V = V_0 \supset V_1 \supset \cdots \supset V_m = 0$ s.t. $\dim V_i / V_{i+1} = 1$.

(Use Lie's Thm repeatedly)

(8.6) Engel's Thm. Let V be a f.d. vector space (over $\mathbb{R}$ or $\mathbb{C}$) $\mathfrak{g} \subset \mathrm{End}\,V$ a Lie subalgebra s.t. $\forall X \in \mathfrak{g}$, $X^N = 0$ for $N \gg 0$.

Then
(1) $\mathfrak{g}$ is nilpotent (as Lie alg.)
(2) $\exists\, v \in V$, $v \neq 0$ s.t. $Xv = 0 \quad \forall X \in \mathfrak{g}$
(3) V admits a flag $V = V_0 \supset V_1 \supset \cdots \supset V_m = 0$
$\dim V_j / V_{j+1} = 1$ and $X(V_j) \subset V_{j+1} \quad \forall X \in \mathfrak{g}$.

Pf. $(2) \Rightarrow (3) \Rightarrow (1)$ is clear. Now assume $\mathfrak{g}$ is nilpotent.
By induction on $\dim \mathfrak{g}$, we prove the existence of a non-zero vector annihilated by all $X \in \mathfrak{g}$. Base case is trivially true.

<u>Claim</u>: $\exists$ a nilpotent ideal $\mathfrak{h} \subset \mathfrak{g}$ s.t. $\dim \mathfrak{g}/\mathfrak{h} = 1$.

Given the claim, write $\mathfrak{g} = \text{Span of} \{X, \mathfrak{h}\}$ and let $V_0 = \{v \in V : Hv = 0 \ \forall H \in \mathfrak{h}\}$

By induction $V_0 \neq 0$. Now $X(V_0) \subset V_0$ since

$$HXv = XHv + [H\,X]v = 0 \quad \forall \begin{matrix} H \in \mathfrak{h} \\ v \in V_0 \end{matrix}$$

$\Rightarrow Xv \in V_0$.

Since X is also nilpotent, we can find $v \in V_0$ s.t. $Xv = 0$.

Proof of the claim. Let $\mathfrak{h} \subset \mathfrak{g}$ be a proper subalg. of max'l dim

Consider $\mathfrak{h} \circlearrowright \mathfrak{g}/\mathfrak{h}$ by $H \cdot X = [H\,X] \pmod{\mathfrak{h}}$.

Every $H \in \mathfrak{h}$ acting on $\mathfrak{g}/\mathfrak{h}$ is nilpotent (even on $\mathfrak{g}$)

As $\dim \mathfrak{h} < \dim \mathfrak{g}$, find $X \in \mathfrak{g} \setminus \mathfrak{h}$ s.t. $H \cdot X = 0 \pmod{\mathfrak{h}}$ $\forall H \in \mathfrak{h}$

i.e. $[H, X] \in \mathfrak{h} \quad \forall H \in \mathfrak{h}$

So $\{\mathfrak{h}, X\}$ span a larger subalg. of $\mathfrak{g}$. By maximality

$\mathfrak{g} = \underset{\substack{\uparrow \\ (\mathbb{R} \text{ or } \mathbb{C})}}{k} \cdot X + \mathfrak{h}$ and $\mathfrak{h}$ is an ideal $\square$

(8.7) <u>Lemma</u> : $\mathfrak{g}$ is nilpotent $\iff$ $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ is.

Pf $\quad [X_1, [X_2, [\ldots [X_\ell, Z]\ldots]]]$

$$= -\,\mathrm{ad}\big([X_2[\ldots [X_\ell, Z]\ldots]]\big)(X_1)$$

$$= -\,[\mathrm{ad}(X_2), [\ldots [\mathrm{ad}(X_\ell), \mathrm{ad}(Z)]\ldots]] \cdot X_1$$

Thus $\mathcal{C}^N \mathfrak{g} = 0 \;\Rightarrow\; \mathcal{C}^N(\mathrm{ad}(\mathfrak{g})) = 0$ and conversely $\square$.

<u>Cor</u> : If $\mathrm{ad}(X) \circlearrowright \mathfrak{g}$ is nilpotent $\forall X \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent

(8.8) Invariant bilinear forms.

Let $\mathfrak{g} \circlearrowright V$ and let $B(\cdot,\cdot) : V \times V \to k$ $(= \mathbb{R}$ or $\mathbb{C})$ be a bilinear form on V. We say B is <u>invariant</u> (for $\mathfrak{g}$-action on V) if $B(X\cdot v, w) + B(v, X\cdot w) = 0$
$\forall X \in \mathfrak{g}$
$v, w \in V$.

<u>Killing Form</u>: $K(X,Y) := \mathrm{Trace}_{\mathfrak{g}}(\mathrm{ad}X \cdot \mathrm{ad}Y)$

is symmetric bilinear form on $\mathfrak{g}$ invariant (w.r.t. adjoint action of $\mathfrak{g}$ on itself), i.e.

$$K([X,Y], Z) + K(Y, [X,Z]) = 0$$

This is because $\mathrm{ad}([X,Y]) \circ \mathrm{ad}Z = \mathrm{ad}X\,\mathrm{ad}Y\,\mathrm{ad}Z - \mathrm{ad}Y\,\mathrm{ad}X\,\mathrm{ad}Z$

has same trace as $\mathrm{ad}Y\,\mathrm{ad}Z\,\mathrm{ad}X - \mathrm{ad}Y\,\mathrm{ad}X\,\mathrm{ad}Z$ $\square$

(8.9) Cartan's Criterion for solvability and semisimplicity.

A. $\mathfrak{g}$ is solvable iff $K(X,Y) = 0 \ \forall X \in \mathfrak{g},\ Y \in [\mathfrak{g},\mathfrak{g}]$

B. $\mathfrak{g}$ is semisimple iff K is non-degenerate.

Remark If $\mathfrak{g}$ is over $\mathbb{R}$, then we can easily check that

$\mathfrak{g}$ is solvable / s-s. $\iff \mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is.

(and K extends / restricts under extn / restriction of scalars).

So for the proof of Cartan's criterion we assume

(without loss of generality) that $\mathfrak{g}$ is a Lie alg. / $\mathbb{C}$.

Proof of A. $\Rightarrow$ B.

If K is degenerate, $\mathrm{Rad}(K) := \left\{ X \in \mathfrak{g} : \begin{array}{c} K(X,Y) = 0 \\ \forall Y \in \mathfrak{g} \end{array} \right\} \neq 0$

$\mathrm{Rad}(K) \subset \mathfrak{g}$ is an ideal in $\mathfrak{g}$ since for $X \in \mathrm{Rad}(K),\ Y \in \mathfrak{g}$

$$K([X,Y], Z) = K(X, [Y,Z]) = 0 \quad \forall Z$$

$\Rightarrow [X,Y] \in \mathrm{Rad}(K)$.

From A., Killing form restricted to K is 0

$\Rightarrow \mathrm{Rad}(K)$ is a solvable ideal of $\mathfrak{g}$

$\Rightarrow \mathfrak{g}$ is NOT semisimple.

Now assume $\mathfrak{g}$ is not semisimple. Then $\exists\ 0 \neq \mathfrak{a} \subset \mathfrak{g}$ an abelian ideal. If we pick a complementary vector space $\mathfrak{s} \subset \mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ then $\forall\ X \in \mathfrak{a},\ Y \in \mathfrak{g}$

$$\mathrm{ad}(X) = \begin{array}{cc} \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} & \begin{array}{c} \mathfrak{a} \\ \mathfrak{s} \end{array} \\ \begin{array}{cc} \mathfrak{a} & \mathfrak{s} \end{array} & \end{array} \qquad \mathrm{ad}(Y) = \begin{array}{cc} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} & \begin{array}{c} \mathfrak{a} \\ \mathfrak{s} \end{array} \\ \begin{array}{cc} \mathfrak{a} & \mathfrak{s} \end{array} & \end{array}$$

$\Rightarrow$ $\mathrm{ad}(X)\,\mathrm{ad}(Y)$ has the form $\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ it has trace 0.

So $K(X,Y) = 0 \quad \forall\ X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$. $\square$

↓ did not cover!

(8.10) Proof of A. Easy part. ↓ lecture 9

Assume $\mathfrak{g}$ is solvable. By Lie's Thm applied to $\mathfrak{g} \overset{\mathrm{ad}}{\circlearrowright} \mathfrak{g}$ we can put $\mathrm{ad}(X)$ in upper Δ^r form simultaneously $\forall\ X \in \mathfrak{g}$. This implies $\mathrm{ad}\,[Y,Z]$ has 0's on the diagonal.

Hence $B(X, [Y,Z]) = 0$.