Lecture 9

(9.0) Recall: we introduced Killing form $K(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ (or $\mathbb{R}$)

$$K(X,Y) := \mathrm{Trace}_{\mathfrak{g}}\left(\mathrm{ad}(X) \circ \mathrm{ad}(Y)\right)$$

Invariance: $\quad K([X,Y],Z) + K(Y,[X,Z]) = 0$

Thm. (Cartan's criterion): (A) $\mathfrak{g}$ is solvable $\Leftrightarrow$ $K(X,Y) = 0$ $\forall X \in \mathfrak{g}$, $Y \in [\mathfrak{g},\mathfrak{g}]$

(B) $\mathfrak{g}$ is semisimple iff K is non-degenerate

Last time we proved (B) assuming (A).

(9.1) Proof of A (easy part) [work over $\mathbb{C}$, see Remark in Lecture 8. section (8.9) on page 8]

($\Rightarrow$). Assume $\mathfrak{g}$ is solvable. Then $[\mathfrak{g},\mathfrak{g}]$ acts on $\mathfrak{g}$ by nilpotent matrices (use Lie's Thm for the adjoint repn.)

So $\mathrm{Tr}\left(\mathrm{ad}(X)\cdot \mathrm{ad}(Y)\right) = 0 \quad \forall X \in \mathfrak{g},\ Y \in [\mathfrak{g},\mathfrak{g}]$ since

$$\mathrm{ad}(X) = \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \text{ and } \mathrm{ad}(Y) = \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}. \qquad \square$$

(9.2) Proof of A. (hard part). For this we need Jordan's decomposition thm and its relatively easier version

Jordan decomposition. Let $X \in M_{n\times n}(\mathbb{C})$. Then $\exists!\ X_s, X_N$ in $M_{n\times n}(\mathbb{C})$ s.t. (i) X_s is diagonalisable, X_N is nilpotent

(ii) $X = X_S + X_N$ (iii) X_S and X_N are polynomials in X ②

(i.e. $\exists\ \mathcal{S}, \mathcal{N} \in \mathbb{C}[T]$ s.t. $X_S = \mathcal{S}(X)$ and $X_N = \mathcal{N}(X)$). In particular

$[X_S, X_N] = 0$.

Moreover $ad(X) \circlearrowright M_{n\times n}(\mathbb{C})$ has s.s. and nilpotent parts $ad(X_S)$, $ad(X_N)$ resp. That is $ad(X)_S = ad(X_S)$

$ad(X)_N = ad(X_N)$

Note: if eigenvalues of X_S are $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ then eigenvalues of $ad(X_S)$ are $\{\alpha_i - \alpha_j\}_{1\le i,j\le n}$.

(9.3) Proof of A. ($\Leftarrow$). We need to prove that $\mathfrak{g}$ is solvable. We will show that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent (which implies $\mathfrak{g}$ is solvable by defn.). Let $X \in [\mathfrak{g},\mathfrak{g}]$. By Engel's Thm and its Corollary ((8.7) of Lecture 8 on page 7), it is enough to prove that $Y = ad(X)$ is nilpotent. ($Y \in M_{n\times n}(\mathbb{C})$, $n = \dim \mathfrak{g}$)

Assume on the contrary, $Y = Y_S + Y_N$, $Y_S \neq 0$.

$$\text{Then } Tr(\overline{Y}_S Y_S) = \sum_{\alpha:\text{ eigenvalue of } Y_S} \overline{\alpha}\alpha > 0 \quad - \quad (*)$$

complex conjugate

$$= Tr(\overline{Y}_S(Y_S + Y_N)) = Tr(\overline{Y}_S Y)$$

(since $\overline{Y}_S Y_N$ is nilpotent)

Since $X \in [\mathfrak{g},\mathfrak{g}]$ we can write $X = \sum_i [X_i', X_i'']$

and consequently $Y = \sum_i [Y_i', Y_i'']$ when $Y_i' = ad(X_i')$.

So $$Tr(\bar{Y}_s Y) = \sum_i Tr(\bar{Y}_s . [Y_i' Y_i''])$$

$$= \sum_i Tr([\bar{Y}_s, Y_i'] \cdot Y_i'')$$ [see the proof of invariance of K in (8.8) of Lecture 8 (p.7)]

$$= \sum_i Tr(\overline{(ad\, Y)_s}(Y_i'] . Y_i'')$$

by (9.2)

$(ad\, Y)_s$ is a polynomial in $ad\, Y$. Thus, so is $\overline{ad\, Y_s}$ ($\mathbb{R}$ coeff.) (* - see page (1))

Now $$(ad\, Y)(Y_i') = ad([X, X_i'])$$

$$(ad\, Y)^2(Y_i') = ad([X[X, X_i']])$$ and so on..

Thus $Tr(\bar{Y}_s Y)$ is a linear combination of terms of the form

$$Tr(\underbrace{ad([X,[X, \ldots [X, X_i'] \ldots]])}_{\in [\mathfrak{g},\mathfrak{g}]}\, ad(X_i'')) = 0$$ by assumption

Hence we contradicted (*) $\Rightarrow Y_s = 0$. So $Y = ad(X)$ is nilpotent $\forall X \in [\mathfrak{g},\mathfrak{g}]$ and thus $[\mathfrak{g},\mathfrak{g}]$ is nilpotent $\square$

(9.4) From now on $\mathfrak{g}$ = a semisimple Lie alg. over $\mathbb{C}$. We record a few consequences of non-degeneracy of K.

__Cor__ (i) Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal. Then $\mathfrak{b} = \mathfrak{a}^{\perp} = \{X \in \mathfrak{g} : K(X,Y)=0 \ \forall Y \in \mathfrak{a}\}$ is another ideal, and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$.

[Proof: $\mathfrak{b}$ is an ideal: let $X \in \mathfrak{g}$, $Y \in \mathfrak{b}$. For every $Z \in \mathfrak{a}$,

$$K([Y,X],Z) = K(Y,\underset{\substack{\cap \\ \mathfrak{a} \text{ as } \mathfrak{a} \text{ is an ideal}}}{[X,Z]}) = 0 \Rightarrow [Y,X] \in \mathfrak{b}.$$

If $\mathfrak{a} \cap \mathfrak{b} \neq (0)$ Then $K \equiv 0$ on $\mathfrak{a} \cap \mathfrak{b} \Rightarrow \mathfrak{a} \cap \mathfrak{b}$ is a solvable ideal contradicting semisimplicity of $\mathfrak{g}$. Finally as K (By A. of Thm (9.0)) is non-deg. $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$].

(ii) $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

[Proof. Since $[\mathfrak{g},\mathfrak{g}]$ is an ideal, its complement $\mathfrak{a}$ must be an ideal as well But then $\mathfrak{a} \simeq \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \leftarrow$ abelian ideal in $\mathfrak{g}$ contradicts s.s. (unless $\mathfrak{a}=0$). $\mathfrak{a}=0$ means $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$ □]

(iii) $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ $\quad$ $\mathfrak{g}_j$'s are simple Lie alg. (min'l ideals of $\mathfrak{g}$).

[Easy consequence of (i) and induction on $\dim \mathfrak{g}$].

(iv) $\mathfrak{g} \xrightarrow[\mathrm{ad}]{} \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ is injective.

[$\mathrm{Ker}(\mathrm{ad}) = Z(\mathfrak{g})$ = center of $\mathfrak{g}$ $:= \{X \in \mathfrak{g} : [X,Y]=0 \ \forall Y \in \mathfrak{g}\}$ is an abelian ideal of $\mathfrak{g}$, hence 0].

(9.5) More generally, let $\mathfrak{g} \circlearrowright V$ faithfully (i.e. $X \cdot v = 0$ $\forall v \in V \Rightarrow X = 0$). Let $B_V(\cdot,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be defined by

$$B_V(x,y) = \mathrm{Trace}_V(\underbrace{x_V \circ y_V})$$

$x_V, y_V \circlearrowright V$ matrices. $\forall x,y \in \mathfrak{g}$

Assume V is simple*

Lemma. (i) B_V is non-degenerate and invariant.

(ii) Let $\{X_i\}$ and $\{X^i\}$ be two bases of $\mathfrak{g}$ s.t.

$$B_V(X_i, X^j) = \delta_{ij}$$

Then
$$\sum_i X_{i,V} X^i_{,V} = \frac{\dim \mathfrak{g}}{\dim V} \cdot \mathrm{Id}_V .$$

(* a representation V is simple / irreducible if 0 and V are the only subrepresentations).

<u>Proof of (i)</u> $B_V([X,Y],Z) + B_V(Y,[X,Z]) = 0$ (same as the one for $K(\cdot,\cdot)$ [Lecture 8, (8.8)].

Now identify $\mathfrak{g}$ with a subalg. of $\mathrm{End}(V)$ and let

$$\mathfrak{a} = \{x \in \mathfrak{g} : \underset{\substack{\| \\ \mathrm{Tr}(xy)}}{B_V(x,y)} = 0 \ \ \forall y \in \mathfrak{g}\} \underset{\text{ideal}}{\subset} \mathfrak{g}$$

via this identification

Using the same argument as in (9.3) we can prove that $[\mathfrak{a}, \mathfrak{a}]$ is nilpotent, hence $\mathfrak{a} \subset \mathfrak{g}$ is solvable $\Rightarrow \mathfrak{a} = 0$, i.e. B_V is non-degenerate.

(ii) We claim that $[X, \sum_{i=1}^{n} X_i X^i] = 0 \quad \forall X \in \mathfrak{g}$. ($n = \dim \mathfrak{g}$)

If true, then $C_V = \sum X_i X^i : V \to V$ is a hom. of $\mathfrak{g}$ representations

Since V is simple $C_V = \lambda \cdot \mathrm{Id}_V$ (let λ be an eigenvalue of C_V; then $\mathrm{Ker}(C_V - \lambda \mathrm{Id}_V)$ is a non-zero $\mathfrak{g}$-subrepn of V, hence $= V$) [Schur's Lemma]

$$\mathrm{Trace}_V(C_V) = \lambda \cdot (\dim V) = \sum_i B_V(X_i X^i) = \dim \mathfrak{g}$$

$$\Rightarrow \lambda = \frac{\dim \mathfrak{g}}{\dim V} \quad \text{as required}$$

Proof of $[X, \sum X_i X^i] = 0$: let
$$[X, X_i] = \sum_j a_i^j X_j$$
$$[X, X^i] = \sum_j b_j^i X^j$$

Then
$$[X, \sum_i X_i X^i] = \sum_i [X X_i] X^i + \sum_i X_i [X X^i]$$
$$= \sum_{i,j} a_i^j X_j X^i + \sum_{i,j} b_j^i X_i X^j = \sum_{i,j} (a_i^j + b_i^j) X_i X^j$$

Now $a_i^j = B([X, X_i], X^j) = -B(X_i, [X X^j]) = -b_i^j$

$$\Rightarrow a_i^j + b_i^j = 0$$

$\square$

(9.6) Thm. Let V be a (f.d.) repn. of $\mathfrak{g}$ and $V' \subset V$ be a subrepn. Then $\exists\ V'' \subset V$ a subrepn. s.t. $V = V' \oplus V''$ as repn. of $\mathfrak{g}$.

Proof. We have $0 \to V' \to V \xrightarrow{pr} V/_{V'} \to 0$ a short exact sequence of $\mathfrak{g}$-repns. We need to find a hom. of $\mathfrak{g}$-repns. $V/_{V'} \xrightarrow{s} V$ s.t. $pr \circ s = \mathrm{Id}_{V/V'}$. Then $V'' = $ Image of s is the required subrepn. of V.

[Aside - review of some definitions. Let A, B be two repns of $\mathfrak{g}$.

$$\mathrm{Hom}_{\mathbb{C}}(A,B) \supset \mathrm{Hom}_{\mathfrak{g}}(A,B) := \left\{ f : A \to B \text{ such that } f(X\cdot a) = X\cdot f(a)\ \forall \begin{smallmatrix} a\in A \\ X\in\mathfrak{g}\end{smallmatrix} \right\}$$

$\mathfrak{g}$ acts naturally on $\mathrm{Hom}_{\mathbb{C}}(A,B)$:

$\forall\ \xi \in \mathrm{Hom}_{\mathbb{C}}(A,B)$, $X \in \mathfrak{g}$, $a \in A$

$$(X\cdot\xi)(a) = X\cdot(\xi(a)) - \xi(X\cdot a)$$

So that

$$\mathrm{Hom}_{\mathfrak{g}}(A,B) = \left\{ \xi \in \mathrm{Hom}_{\mathbb{C}}(A,B) \text{ such that } \begin{smallmatrix} X\cdot\xi = 0 \\ \forall\, X\in\mathfrak{g}\end{smallmatrix} \right\}$$]

Continuing with the proof, pick any $l \in \mathrm{Hom}_{\mathbb{C}}(V/_{V'}, V)$ s.t. $pr \circ l = \mathrm{Id}_{V/V'}$. We will modify l so that it lands in $\mathrm{Hom}_{\mathfrak{g}}(V/_{V'}, V)$ still satisfying $pr \circ l = \mathrm{Id}_{V/V'}$.

Consider
$$\begin{array}{ccc} f: \mathfrak{g} & \longrightarrow & \mathrm{Hom}_{\mathbb{C}}(V/V', V) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ X & \longmapsto & X\cdot \ell \end{array}$$
We claim that

$f(X) \in \mathrm{Hom}_{\mathbb{C}}(V/V', V')$. i.e. $\mathrm{pr}\circ f(X) = 0$ $\forall X \in \mathfrak{g}$. This is

clear since $(\forall a \in V/V')$ $\mathrm{pr}\circ f(X)(a) = (\mathrm{pr}\circ(X\cdot \ell))(a) = \mathrm{pr}\left((X\cdot\ell)(a)\right)$

$= \mathrm{pr}\left(X\cdot(\ell(a)) - \ell(X\cdot a)\right) = X\cdot(\mathrm{pr}(\ell(a))) - (\mathrm{pr}\circ\ell)(X\cdot a)$

(since $\mathrm{pr}: V \to V/V'$ is a hom of $\mathfrak{g}$-repns.)

$= X\cdot a - X\cdot a = 0$

(since $\mathrm{pr}\circ\ell = \mathrm{Id}_{V/V'}$).

Moreover $f([X,Y]) = X\cdot f(Y) - Y\cdot f(X)$ (after all $f(X) = X\cdot\ell$)

Crucial Lemma: Let R be a repn. of $\mathfrak{g}$ and assume we have a $\mathbb{C}$-linear map $f: \mathfrak{g} \to R$ s.t.

$$f([X,Y]) = X\cdot f(Y) - Y\cdot f(X).$$

Then $\exists\ r \in R$ s.t. $f(X) = X\cdot r$

Apply it to $R = \mathrm{Hom}_{\mathbb{C}}(V/V', V')$ and
$$\begin{array}{ccc} f: \mathfrak{g} & \to & \mathrm{Hom}(V/V', V') \\ X & \longmapsto & X\cdot\ell \end{array}$$

to find $m \in \mathrm{Hom}_{\mathbb{C}}(V/V', V')$ s.t. $X\cdot\ell = X\cdot m$

i.e. $X\cdot(\ell - m) = 0$ $(\mathrm{pr}\circ(\ell-m) = \mathrm{pr}\circ\ell = \mathrm{Id})$

i.e. $\ell - m \in \mathrm{Hom}_{\mathfrak{g}}(V/V', V)$. $\square$

(9.7) Proof of Crucial Lemma.

Case 1. $\mathcal{R}$ is irreducible and $\mathfrak{g} \to \mathrm{End}(\mathcal{R})$ is injective.

Use Lemma (9.5). We claim $r = \frac{1}{\lambda} \sum_i X_i \cdot f(X^i) \in \mathcal{R}$.

We need to prove that $f(X) = X \cdot r \quad (\forall X \in \mathfrak{g})$.

$$\lambda\left(f(X) - X \cdot r \right) = \sum_i X_i \cdot (X^i \cdot f(X)) - \sum_i X \cdot (X_i \cdot f(X^i))$$

$$= - \sum_i \underbrace{[X, X_i]}_{\sum_j a_i^j X_j} \cdot f(X^i) + X_i \cdot f(\underbrace{[X, X^i]}_{\sum \cancel{b_i^j}\, b_j^i X^j}) \quad \left(\text{by assumption on } f\right)$$

$$= - \sum_{i,j} (a_i^j + b_i^j)\, X_j \cdot f(X^i) = 0$$

Case 2. $\mathcal{R}$ is irreducible. Let $\mathfrak{k} = \{ X \in \mathfrak{g} : X \cdot a = 0 \ \forall a \in \mathcal{R} \}$.

$\mathfrak{k} \subset \mathfrak{g}$ is an ideal, hence $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$ (Cor 9.4 (i) p. 4)

(both $\mathfrak{k}$ and $\mathfrak{b}$ s.s.). By Case 1, $\exists\, r \in \mathcal{R}$ s.t. $f(X) = X \cdot r$ $\forall X \in \mathfrak{b}$.

Now $f([Y, Z]) = Y \cdot f(Z) - Z \cdot f(Y) = 0 \quad \forall\, Y, Z \in \mathfrak{k}$.

Since $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}]$ (as $\mathfrak{k}$ is semisimple Cor 9.4 (ii))

$0 = f(X) = X \cdot r \quad \forall X \in \mathfrak{k}$.

Case 3. R arbitrary. By induction on $\dim R$.

If R is not irreducible, we have a subrepn $R' \subsetneq R$ $(0 \neq R')$

Apply induction hypothesis to

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & R/R' \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ X & \longmapsto & pr(f(X)) \end{array}$$

where

$pr: R \to R/R'$. We find $r_1 \in R/R'$ s.t. $pr(f(X)) = X \cdot r_1$

Choose any $\tilde{r}_1 \in R$ s.t. $pr(\tilde{r}_1) = r_1$. Then we have

$$pr(f(X)) = pr(X \cdot \tilde{r}_1) \quad \text{since } pr \text{ is a hom of } \mathfrak{g}\text{-repns.}$$

Apply induction again to

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & R' \\ X & \longmapsto & f(X) - X \cdot \tilde{r}_1 \end{array}$$

and find

$r_2 \in R'$ s.t $f(X) - X \cdot \tilde{r}_1 = X \cdot r_2$. Then $r = r_2 + \tilde{r}_1$

□

(9.8) Explanation of $(*)$ on page 3. We know $\exists\, s(T) \in \mathbb{C}[T]$ s.t.

$$s(X) = X_s = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

(in a suitable basis)

. Take $f(t) \in \mathbb{C}[t]$ s.t.

$f(0) = 0$ and $f(\lambda_i) = \overline{\lambda_i}$ $(1 \leq i \leq n)$

Then $(f \circ s)(X) = \overline{X_s}$.

(e.g. if $\alpha_1, \ldots, \alpha_k$ are distinct complex #'s

$$\sum_i \overline{\alpha_i} \prod_{j \neq i} \frac{(t - \alpha_j)}{\alpha_i - \alpha_j} \quad \text{maps } \alpha_i \mapsto \overline{\alpha_i} \quad \forall\, 1 \leq i \leq k)$$