Lecture 10

(10.0) Recall that last time we proved the complete reducibility theorem (Thm (9.6) of Lecture 9, page 7): $\mathfrak{g}$ semisimple Lie alg $/\mathbb{C}$, $\mathfrak{g} \circlearrowright V$ f.d. repn. If V' is a subrepn of V then $\exists\, V'' \subset V$ a ~~supr~~ subrepn. s.t. $V = V' \oplus V''$.

(10.1) Cor. Every derivation of $\mathfrak{g}$ is inner. That is, let $D : \mathfrak{g} \to \mathfrak{g}$ be a linear map s.t. $D([x,y]) = [D(x), y] + [x, D(y)]$ $\forall\, x, y \in \mathfrak{g}$. Then $\exists\, x_0 \in \mathfrak{g}$ s.t. $D = ad(x_0)$.

Proof. Set $V = \mathbb{C}\cdot\partial + \mathfrak{g}$ with Lie alg. str. $[\partial, x] = D(x)$ and $\mathfrak{g} \subset V$ is a subalgebra. Thus $\mathfrak{g} \circlearrowright V$ and $V' = \mathfrak{g}$ is a subrepn. Thm (9.6) $\Rightarrow$ $\exists\; v = (\partial, x_0)$ s.t. $V = \mathbb{C}v \oplus \mathfrak{g}$ as repn. of $\mathfrak{g}$. Note that one-dim'l repn. of $\mathfrak{g}$ has to be trivial, since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ (Pf. Let $\mathfrak{g} \circlearrowright \mathbb{C}$ be a 1-dim'l repn. Then $\forall X \in \mathfrak{g}$, $X \cdot \mathbb{1} = \lambda(X)\cdot \mathbb{1}$ for some scalar $\lambda(X)$. Thus $[X, Y]\cdot \mathbb{1} = 0$ $\forall X, Y \in \mathfrak{g}$ but $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ - see (9.4) (ii) of Lecture 9 page 4).

So $\forall\; X \in \mathfrak{g}$ $\quad [X, \partial + x_0] = 0 = -D(X) - [x_0, X]$

i.e. $D = ad(-x_0)$ $\square$

Note: We already know that $ad : \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ is injective and $\mathrm{Image}(ad) \subset \mathrm{Der}(\mathfrak{g})$. Corollary above says that $\mathfrak{g} \simeq \mathrm{Der}(\mathfrak{g})$ via ad.

(10.2) Representations of $sl_2(\mathbb{C})$

$sl_2(\mathbb{C})$ = Lie algebra of 2×2 matrices of trace 0. Set

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ a basis of $sl_2(\mathbb{C})$.

Easy check: $\Big(\underset{\substack{\uparrow \\ \text{obviously}}}{[h,h]=0}\Big)$ $[h,e]=2e$ $[h,f]=-2f$ $[e,f]=h$

Killing form $K(h,h)=8$ $K(e,f)=4$. All other 0.

So $sl_2(\mathbb{C})$ is a semisimple Lie algebra

In fact one can prove directly that $sl_2\mathbb{C}$ has no nontrivial proper ideals. Hence it is simple.

Proof Let $\mathfrak{a} \subset sl_2\mathbb{C}$ be a non-zero ideal. Let $\overset{z}{\overset{\shortparallel}{\alpha e + \beta h + \gamma f}} \in \mathfrak{a}$

We may assume $\alpha \neq 0$. (if $\substack{\alpha=0 \\ \beta\neq 0}$, $[e,z]\in\mathfrak{a}$ with coeff of e being $-2\beta\neq 0$
if $\substack{\alpha=\beta=0 \\ \gamma\neq 0}$, $[e,[e,z]]$ has coeff of $e \neq 0$)

Then $[f,[f,z]]\in\mathfrak{a}$ and $[f,[f,z]] = 2\alpha f \in \mathfrak{a} \Rightarrow f\in\mathfrak{a}$

$\Rightarrow h=[e,f]\in\mathfrak{a}$ $\Rightarrow$ $e=\frac{1}{2}[h,e]\in\mathfrak{a}$. So $\mathfrak{a}=sl_2(\mathbb{C})$ □

Let V be a f.d. irreducible repn. of sl_2. Let $v\in V$ $(0\neq v)$ be an eigenvector for h, say $h\cdot v=\lambda v$ $(\lambda\in\mathbb{C})$.

Then $h\cdot(e\cdot v) = e\cdot(h\cdot v)+[h,e]\cdot v = (\lambda+2)(e\cdot v)$

$\Rightarrow$ $h\cdot(e^2\cdot v) = (\lambda+4)(e^2\cdot v)$ and so on.

So $v, e.v, e^2.v, \ldots$ are all eigenvectors of h with distinct eigenvalues $(\lambda, \lambda+2, \lambda+4, \ldots)$. Hence they are linearly independent and by finite-dimensionality of V, $\exists\, k$ s.t. $e^k.v \neq 0$ but $e^{k+1}.v = 0$.

Replace $v_0 := e^k.v$ $\quad \mu := \lambda + 2k$. $\quad$ So $e.v_0 = 0$

Define $v_j = f^j . v_0$ $(j \geq 0)$. Same calculation as above shows that $h.v_j = (\mu - 2j)\, v_j$. Now

$$[e, f^j] = \sum_{i=0}^{j-1} f^i\, [e,f]\, \cancel{f}\, f^{j-i-1} = \sum_{i=0}^{j-1} f^i\, h\, f^{j-i-1}$$

Thus
$$e.v_j = e.(f^j.v_0) = \cancelto{0}{f^j.(e.v_0)} + [e, f^j].v_0$$
$$= \sum_{i=0}^{j-1} f^i . h . v_{j-i-1} = \left[\sum_{i=0}^{j-1} (\mu - 2(j-i-1)) \right] v_{j-1}$$
$$= [\, j \cdot \mu - 2j(j-1) + j(j-1)\,]\, v_{j-1}$$
$$= j\, [\mu - j + 1]\, v_{j-1}$$

$\Rightarrow \{v_0, v_1, v_2, \ldots\}$ span a subrepn. of V, hence $= V$.

If ℓ is the smallest subscript s.t. $v_\ell = 0$ (by finite-dimensionality of V)

then $0 = e.v_\ell = \ell(\mu - \ell + 1)\, v_{\ell-1}$

$\Rightarrow \mu = \ell - 1 \in \mathbb{Z}_{\geq 0}$.

Thus we have $V = $ Span of $\{v_0, v_1, \dots, v_r\}$ $(r = \ell - 1)$

$h \cdot v_j = (r - 2j) v_j$ $\quad f \cdot v_j = v_{j+1}$ $\quad e \cdot v_j = j(r-j+1) v_{j-1}$

(convention : $v_{r+1} = v_{-1} = 0$). $\quad \dim V = r+1$ $\quad (r \in \mathbb{Z}_{\geq 0})$

And these are all irreducible f.d. repns. of $\mathfrak{sl}_2(\mathbb{C})$.

(10.3) Cor Let V be a f.d. repn. of $\mathfrak{sl}_2(\mathbb{C})$. If $v \in V$ is an eigenvector for h with eigenvalue μ s.t. $ev = 0$ then $\mu \in \mathbb{Z}_{\geq 0}$.

An eigenvector for h is usually called a <u>weight vector</u>. A weight vector (of wt. μ) is a <u>highest weight vector</u> if $e.v = 0$.

(10.4) Classification of semisimple Lie algebras - Root Systems :

Defn. Let V be a f.d. $\mathbb{R}$-vector space together with $(\cdot,\cdot) : V \times V \to \mathbb{R}$ a positive definite, symmetric bilinear form. A (finite) <u>root system</u> is a finite set $R \subset V \setminus \{0\}$ s.t.

(i) R spans V (note: we are not assuming R is a basis)

(ii) $\alpha \in R$ and $c\alpha \in R \Rightarrow c = \pm 1$

(iii) $\forall\, \alpha, \beta \in R$, $\quad \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$. Define $s_\alpha \in GL(V)$

$$s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)} \cdot \alpha$$

(iv) $\forall\, \alpha, \beta \in R$, $\quad s_\alpha(\beta) \in R$.

$$\begin{tikzcd}
\text{Semi-Simple Lie Algebras} \arrow[r, leftrightarrow, "\text{I}"'] & \text{Root systems} \arrow[r, leftrightarrow, "\text{II}"'] & \text{Cartan Matrices (or Dynkin diagrams)} \\
\text{Simple} \arrow[u, phantom, "\cup" description] \arrow[rr, leftrightarrow, bend right, "\text{III}"'] & \text{Irreducible} \arrow[u, phantom, "\cup" description] & \text{connected Dynkin diagrams} \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

Correspondence $\xrightarrow{\text{I}}$: Let $\mathfrak{g}$ be a semisimple Lie alg / $\mathbb{C}$.

I a). $\exists$ an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ s.t. $\mathrm{ad}_{\mathfrak{g}}(x)$ is semisimple $\forall x \in \mathfrak{h}$ and $\mathfrak{h}$ is max'l. [Cartan Subalgebra]

Thus $$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} \underbrace{\mathfrak{g}_\alpha}_{\text{root space}} \qquad \left(\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} : [h,x] = \alpha(h)x \ \forall h \in \mathfrak{h} \} \right)$$

I b). $\dim \mathfrak{g}_\alpha = 1$

$$V = \mathbb{R}\text{-span of } \{\alpha : \mathfrak{g}_\alpha \neq 0\} \subset \mathfrak{h}$$
$$\cup$$
$$R = \{\alpha \neq 0 : \mathfrak{g}_\alpha \neq 0\} \quad \text{roots}$$

$(\cdot,\cdot)$ on V = Killing form

Then R is a (finite) root system.

[Independence of choices: any two $\mathfrak{h}_1, \mathfrak{h}_2$ of I a) are conjugate to each other via an element $\eta \in \mathrm{Aut}_{\mathrm{LieAlg}}(\mathfrak{g})$]

Correspondence (II) $\longrightarrow$

II a). $W = \langle s_\alpha : \alpha \in R \rangle \subset GL(V)$ is a finite group (since it is discrete and is a subgroup of $O(V)$ - preserving $(\cdot,\cdot)$)

II b). $\exists\,!$ (upto W-conjugation) <u>base</u> of R, i.e. a subset $B \subset R$ s.t.

- B is a basis of V, $B = \{\alpha_1, \dots, \alpha_\ell\}$
- $\forall\, \alpha \in R$, $\alpha = \sum c_i \alpha_i$ then $c_i \in \mathbb{Z}_{\geq 0}$ $\forall i$ or $c_i \in \mathbb{Z}_{\leq 0}$ $\forall i$

Set $A = \left(a_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right)_{1 \leq i,j \leq \ell}$ $\leftarrow$ Cartan matrix

Note $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\leq 0}$ are of the form $\{0, -1, -2, -3\}$:

2×2 possibilities $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$\Gamma(A)$ (Dynkin diagram) : Vertices : $1, \dots, \ell$

Edges

$i \quad j$	$a_{ij} = 0$
$i - j$	$a_{ij} = -1 = a_{ji}$
$i \Leftarrow j$	$a_{ij} = -2 \quad a_{ji} = -1$
$i \Lleftarrow j$	$a_{ij} = -3 \quad a_{ji} = -1$

These are classified : $A_n, B_n, C_n, D_n, E_{6,7,8}, F_4, G_2$:

A_n : $1 - 2 - 3 - \cdots - n-1 - n$ $(n \geq 1)$

B_n : $1 - 2 - 3 \cdots - n-1 \Leftarrow n$ $(n \geq 2)$

C_n : $1 - 2 - 3 \cdots - n-1 \Rightarrow n$ $(n \geq 3)$

D_n : $(n \geq 4)$

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & & n-1 \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \cdots \arrow[r, dash] & n-2 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & n
\end{tikzcd}$$

$E_{6,(7(8))}$

$$\begin{tikzcd}[column sep=small]
1 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 5 \arrow[r, dash] & 6\ (- 7\ (- 8)) \\
 & & 4 & &
\end{tikzcd}$$

F_4 $1 - 2 \Leftarrow 3 - 4$

G_2 $1 \Lleftarrow 2$

Correspondence $\xrightarrow{\text{III}}$ Let $A = (a_{ij})_{1 \leq i,j \leq \ell}$ be a Cartan matrix.

Pick $d_1, \dots, d_\ell \in \mathbb{Z}_{>0}$ s.t. $\gcd(d_1, \dots, d_\ell) = 1$ and $d_i a_{ij} = d_j a_{ji}$

Set $\mathfrak{h} = \text{Span of } \{h_1, \dots, h_\ell\}$ (ℓ-dim'l $\mathbb{C}$-v.s.)

$\mathfrak{g}$ = Lie alg. generated by $\{h_i, e_i, f_i\}_{i=1,\dots,\ell}$ subject to:

$$[h_i, h_j] = 0 \qquad [h_i, e_j] = a_{ij} e_j \qquad [h_i, f_j] = -a_{ij} f_j$$

$$[e_i, f_j] = \delta_{ij} h_i$$

$$\forall\, i \neq j \qquad \operatorname{ad}(e_i)^{1-a_{ij}} \cdot e_j = 0 = \operatorname{ad}(f_i)^{1-a_{ij}} \cdot f_j$$