Lecture 11

(11.0) Recall: for a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ we proved complete reducibility theorem and the fact that all derivations of $\mathfrak{g}$ are inner (Thm (9.6) of Lecture 9, page 7; and Cor (10.1) of Lecture 10, page 1).

Also we classified irreducible f.d. repns of $sl_2(\mathbb{C})$ (section (10.2)) and proved that if $sl_2(\mathbb{C}) \circlearrowright V$ a f.d. repn and $v \in V$ is such that ($v \neq 0$) $h.v = \lambda v$ ($\lambda \in \mathbb{C}$) and $e.v = 0$, then $\lambda \in \mathbb{Z}_{\geq 0}$.

(11.1) Prop. Let $\mathfrak{a}$ be a nilpotent Lie algebra and $\mathfrak{a} \circlearrowright V$ a f.d. repn.
For $\lambda \in \mathfrak{a}^*$ define
$$\underset{\substack{\uparrow}}{V_\lambda} = \left\{ v \in V : (X - \lambda(X).\mathrm{Id})^N . v = 0 \ \text{ for each } X \in \mathfrak{a}, \ N \gg 0 \text{ (depending on } X \,\&\, v) \right\}$$
generalized eigenspace of eigenvalue λ.

Then $V = \bigoplus_{\lambda \in \mathfrak{a}^*} V_\lambda$ as repns of $\mathfrak{a}$.

Proof. The assertion is trivially true if $V = V_\lambda$ for some $\lambda \in \mathfrak{a}^*$.
In general, we choose $X \in \mathfrak{a}$ and consider the Jordan canonical form of X acting on V, such that there are at least 2 blocks

$$X \text{ acting on } V = \begin{bmatrix} \begin{matrix} a_1 & 1 & 0 \\ & \ddots & 1 \\ 0 & & a_1 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} a_s & 1 & 0 \\ & \ddots & 1 \\ 0 & & a_s \end{matrix} \end{bmatrix}$$

For a an eigenvalue of X, let V_a = generalized eigenspace of X w/ eigenvalue a
$$V_a = \{ v \in V : (X - a)^N v = 0 \text{ for } N \gg 0 \}$$

Claim: $\mathfrak{a}$ preserves each V_a.

Hence $V = \bigoplus_{j=1}^{s} V_{a_j}$ is a decomposition of V into repns. of $\mathfrak{a}$.
Thus the assertion will follow by induction

Proof of Claim. Let us assume $a = 0$ (o/w replace X by $X - a \cdot \mathrm{Id}$).

For $v \in V_0$, $Y \in \mathfrak{a}$, we have the following formula (easily proved by induction)

$$X^k \cdot (Y \cdot v) = \sum_{p=0}^{k} \binom{k}{p} \left(\mathrm{ad}(X)^p \cdot Y\right) \cdot \left(X^{k-p} \cdot v\right) . \quad (*)$$

As $\mathfrak{a}$ is nilpotent $\exists N_1$ s.t. $\mathrm{ad}(X)^{N_1} \cdot Y = 0$. By defn $\exists N_2$ s.t. $X^{N_2} \cdot v = 0$. Then take $k \geq N_1 + N_2$. The formula above implies that $X^{N_1 + N_2} \cdot (Y \cdot v) = 0 \Rightarrow Y \cdot v \in V_0$ ($\forall Y \in \mathfrak{a}$, $v_0 \in V_0$). $\square$

(11.2) For $\mathfrak{g}$ an arbitrary Lie algebra over $\mathbb{C}$, and $x \in \mathfrak{g}$, define

$$\mathfrak{g}_0(x) = \{ Y \in \mathfrak{g} : \mathrm{ad}(X)^N \cdot Y = 0 \text{ for } N \gg 0 \}$$

(generalized eigenspace of $\mathrm{ad}(X) \circlearrowright \mathfrak{g}$ w/ eigenvalue 0)

Prop - Defn. $\exists\, \ell \geq 1$ s.t. $\forall\, x \in \mathfrak{g}$, $\dim \mathfrak{g}_0(x) \geq \ell$. ~~Moreover,~~ and for some $x \in \mathfrak{g}$, $\dim \mathfrak{g}_0(x) = \ell$.

We define $\mathrm{rank}(\mathfrak{g}) = \ell$. Note that $x \in \mathfrak{g}_0(x) \Rightarrow \ell \geq 1$ $\forall \mathfrak{g}$.

As we will see in the proof $\{ x : \dim \mathfrak{g}_0(x) = \ell \}$ is a dense open subset of $\mathfrak{g}$.

Proof. Define polynomial functions $P_{N-1}, P_{N-2}, \dots : \mathfrak{g} \to \mathbb{C}$ (here $N = \dim \mathfrak{g}$) by

Characteristic polynomial of $\mathrm{ad}(x) \circlearrowright \mathfrak{g}$

$$= \det(\lambda \, \mathrm{Id}_{\mathfrak{g}} - \mathrm{ad}(x)) = \lambda^N + P_{N-1}(x)\lambda^{N-1} + \cdots$$

Let ℓ be smallest subscript s.t. $P_\ell \not\equiv 0$ (as a polynomial fn. on $\mathfrak{g}$).

Note: for $y \in \mathfrak{g}$, $\dim(\mathfrak{g}_0(y)) = \min\{p : P_p(y) \neq 0\}$

This proves the proposition and $\{x : \dim \mathfrak{g}_0(x) = \ell\} = \{x : P_\ell(x) \neq 0\}$ is a dense open subset of $\mathfrak{g}$. $\square$

(11.3) Continuing with general set up, let $x \in \mathfrak{g}$ be s.t. $P_\ell(x) \neq 0$ ($\ell = \mathrm{Rank}(\mathfrak{g})$).

Lemma. (1) $\mathfrak{g}_0(x)$ is a nilpotent Lie subalgebra of $\mathfrak{g}$.

(2) if $y \in \mathfrak{g}$ is s.t. $[y, \mathfrak{g}_0(x)] \subset \mathfrak{g}_0(x)$ then $y \in \mathfrak{g}_0(x)$.

(3) $\mathfrak{g}_0(x)$ is the unique such subalgebra containing x

Proof. Let us write $\mathrm{ad}(x) \circlearrowright \mathfrak{g}$ in its Jordan canonical form

so $$\mathfrak{g} = \underbrace{\mathfrak{g}_0(x)}_{\substack{\text{generalized} \\ \text{eigenspace w/ e.v.} = 0}} \oplus \bigoplus_{a \neq 0} \mathfrak{g}_a(x)$$

$\mathfrak{g}_a(x)$: corr. to eigenvalue $a \neq 0$

$$\mathfrak{g}_*(x) := \bigoplus_{a \neq 0} \mathfrak{g}_a(x)$$

Formula ⊛ of page 2 $\Rightarrow$ $\forall\, y \in \mathfrak{g}_0(x)$, $z \in \mathfrak{g}_a(x)$

$[y, z] \in \mathfrak{g}_a(x)$

In particular $\mathfrak{g}_0(x)$ is a subalgebra of $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{g}_0(x) \oplus \mathfrak{g}_*(x) \quad \text{as repns of } \mathfrak{g}_0(x) \text{ via ad.}$$

Now let $S = \{ y \in \mathfrak{g}_0(x) : \mathrm{ad}(y) : \mathfrak{g}_*(x) \to \mathfrak{g}_*(x) \text{ is iso.} \}$

$R = \{ y \in \mathfrak{g}_0(x) : \mathrm{ad}(y) : \mathfrak{g}_0(x) \to \mathfrak{g}_0(x) \text{ is NOT nilpotent} \}$

Both S and R are open (& dense, if non-empty) subsets of $\mathfrak{g}_0(x)$.
$S \neq \phi$ because $x \in S$. If $R \neq \phi$, then we will have $y \in S \cap R$
and $\mathfrak{g}_0(y) \subsetneq \mathfrak{g}_0(x)$ will have $\dim < l$. This is a contradiction
implying that $R = \phi$, i.e., every $\mathrm{ad}(y) \circlearrowright \mathfrak{g}_0(x)$ ($y \in \mathfrak{g}_0(x)$) is
nilpotent. By Engel's Thm (8.6) page 5 of Lecture 8, $\mathfrak{g}_0(x)$ is
nilpotent Lie subalgebra of $\mathfrak{g}$. (1) is proved. (2) is obvious.

Later $\left\{ \text{Let us write } \mathfrak{g} = \mathfrak{g}_0(x) \oplus \bigoplus_{\lambda \in \mathfrak{g}_0(x)^* \setminus \{0\}} \mathfrak{g}_\lambda(x) \text{ according to prop. (11.1)} \right\}$

(3). Let $\mathfrak{h}$ be a nilpotent Lie subalgebra of $\mathfrak{g}$ s.t. $x \in \mathfrak{h}$ &
$[Y, \mathfrak{h}] \subset \mathfrak{h} \Rightarrow Y \in \mathfrak{h}$. Then $\mathfrak{h} \subset \mathfrak{g}_0(x)$ because
$\mathrm{ad}(x) \circlearrowright \mathfrak{h}$ is nilpotent. Assume $\mathfrak{g}_0(x)/\mathfrak{h}$ is non-zero.
Since $\mathfrak{h} \subset \mathfrak{g}_0(x) \underset{\mathrm{ad}}{\circlearrowright} \mathfrak{g}_0(x)$ is nilpotent, $\mathfrak{h} \circlearrowright \mathfrak{g}_0(x)/\mathfrak{h}$ is
as well. Hence we can find $0 \neq \overline{Y} \in \mathfrak{g}_0(x)/\mathfrak{h}$ s.t. $H \cdot \overline{Y} = 0$ $\forall H \in \mathfrak{h}$
i.e. $Y \in \mathfrak{g}_0(x) \setminus \mathfrak{h}$ and $[Y, \mathfrak{h}] \subset \mathfrak{h}$. But this contradicts

our assumption on $\mathfrak{h}$. Hence $\mathfrak{h} = \mathfrak{g}_0(x)$. $\square$

(11.4) Consequences of Lemma (11.3):

For simplicity of notations, write $\mathfrak{h} = \mathfrak{g}_0(x)$ and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha \qquad \text{according to Prop. (11.1)}$$

We have (from the proof of Lemma (11.3))

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \qquad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \qquad [\mathfrak{h}, \mathfrak{g}_\alpha] \subseteq \mathfrak{g}_\alpha$$

Hence if $X \in \mathfrak{g}_\alpha$ $(\alpha \neq 0)$ then $\mathrm{Trace}(\mathrm{ad}(X)) = 0$. In particular if

$X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $\alpha + \beta \neq 0$, then $K(X, Y) = 0$.

Cor. • K restricted to $\mathfrak{g}_\alpha \times \mathfrak{g}_\beta \equiv 0$ if $\alpha + \beta \neq 0$

~~(11.5) Now assume $\mathfrak{g}$ is semisimple~~

- for $\alpha \neq 0$, $X \in \mathfrak{g}_\alpha$, $\mathrm{ad}(X) \circlearrowright \mathfrak{g}$ is nilpotent

(since $\mathrm{ad}(X)^k : \mathfrak{g}_\beta \to \mathfrak{g}_{\beta + k\alpha}$ eventually 0 vector sp.)

(11.5) For general $\mathfrak{g}$, a nilpotent subalgebra $\mathfrak{h}$ satisfying

$$[Y, \mathfrak{h}] \subset \mathfrak{h} \Rightarrow Y \in \mathfrak{h}$$

is called a Cartan subalgebra. Lemma (11.3) proves the existence of Cartan subalgebras. Now we can prove uniqueness

Lemma. Let $\mathfrak{h}_1, \mathfrak{h}_2$ be two Cartan subalgebras of $\mathfrak{g}$. Then

$$\exists\ \eta \in \mathrm{Aut}_{\text{Lie Alg}}(\mathfrak{g}) \quad \text{s.t.} \quad \eta(\mathfrak{h}_1) = \mathfrak{h}_2.$$

Proof. Let

$$\left.\begin{aligned} \mathfrak{g} &= \mathfrak{h}_1 \oplus \mathfrak{g}_{\lambda_1} \oplus \cdots \oplus \mathfrak{g}_{\lambda_K} \\ &= \mathfrak{h}_2 \oplus \mathfrak{g}_{\mu_1} \oplus \cdots \oplus \mathfrak{g}_{\mu_\ell} \end{aligned}\right\} \text{decompositions according to Prop (11.1)}$$

Each one defines a poly map $\eta_i : \mathfrak{g} \to \mathfrak{g}$ as follows

$$\left.\begin{aligned} \eta_1(h, x_1, \ldots, x_K) &= \exp(\mathrm{ad}(x_1)) \cdots \exp(\mathrm{ad}(x_K)) \cdot h \\ \eta_2(h', y_1, \ldots, y_\ell) &= \exp(\mathrm{ad}(y_1)) \cdots \exp(\mathrm{ad}(y_\ell)) \cdot h' \end{aligned}\right\} \begin{array}{l} \text{Poly. since} \\ \mathrm{ad}(x_i) \\ \mathrm{ad}(y_j) \\ \text{are nilpot.} \\ \text{- Cor. (11.4)} \end{array}$$

Let $\mathfrak{h}_1^{reg} = \mathfrak{h}_1 \setminus \bigcup_{j=1}^{K} \mathrm{Ker}(\lambda_j)$ (similarly $\mathfrak{h}_2^{reg}$)

Easy check: for $h_i \in \mathfrak{h}_i^{reg}$, $T_{h_i}\eta_i : \mathfrak{g} \to \mathfrak{g}$ is surjective

$$\Rightarrow \eta_1(\mathfrak{h}_1^{reg} \times \mathfrak{g}_{\lambda_1} \times \cdots \times \mathfrak{g}_{\lambda_K}) \cap \eta_2(\mathfrak{h}_2^{reg} \times \mathfrak{g}_{\mu_1} \times \cdots \times \mathfrak{g}_{\mu_\ell}) \neq \phi$$

(both contain dense open subset)

i.e. $\exists\ w_1, w_2 \in \mathrm{Aut}_{\text{Lie Alg}}(\mathfrak{g})$, $h_1, h_2 \in \mathfrak{h}_1^{reg}, \mathfrak{h}_2^{reg}$ s.t

$$w_1(h_1) = w_2(h_2) \Rightarrow h_1 = w_1^{-1} w_2(h_2)$$

Now the result follows since $\mathfrak{h}_i = \mathfrak{g}_0(h_i)$ $(i=1,2)$

□

(11.6) Now assume $\mathfrak{g}$ is semisimple. Some easy consequences of the results of (11.4) and (11.5) are

(i) $K|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate. $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is non-degenerate

(ii) Write $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus 0} \mathfrak{g}_\alpha$ (according to Prop (11.1))

If $\mathfrak{h}$ contains h s.t. $\alpha(h) = 0$ $\forall \alpha \in R$ $(:= \{\tilde{\alpha} \in \mathfrak{h}^* : \mathfrak{g}_{\tilde{\alpha}} \neq 0\})$

then $h = 0$. This is because

$$K(h_1, h_2) = \sum_{\alpha \in R} (\dim \mathfrak{g}_\alpha)\, \alpha(h_1)\, \alpha(h_2)$$

so if $\alpha(h) = 0$ $\forall \alpha \in R$, we get $K(h, h') = 0$ $\forall h' \in \mathfrak{h}$. By non-degeneracy of $K|_{\mathfrak{h}\times\mathfrak{h}}$, $h = 0$.

Hence R spans $\mathfrak{h}^*$.

(iii) Every $h \in \mathfrak{h}$ acts semisimply on $\mathfrak{g}$ (i.e. $\mathrm{ad}_{\mathfrak{g}}(h)$ is s.s.)

Proof. Let $S + N = \mathrm{ad}(h)$ be Jordan decomposition. Then $S \cdot x = \alpha(h) x$ $\forall x \in \mathfrak{g}_\alpha$. Hence $S \circlearrowright \mathfrak{g}$ is a derivation:

$$\forall\, x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_\beta : \quad S([x,y]) = (\alpha+\beta)(h) \cdot [x,y] = (\alpha(h) + \beta(h))\,[x,y] = [S\cdot x, y] + [x, S\cdot y]$$

$\Rightarrow$ we can find $h_s \in \mathfrak{g}$ s.t. $S = \mathrm{ad}(h_s)$ [every derivation is inner!]

But $[h_s, x] = 0$ $\forall x \in \mathfrak{h}$ $\Rightarrow$ $h_s \in \mathfrak{h}$.

and $N = \mathrm{ad}(h - h_s)$, acts as 0 on each $\mathfrak{g}_\alpha$ so $h - h_s = 0$ hence $\mathrm{ad}(h)$ is semisimple