Lecture 12

(12.0) Recall last time we proved existence of Cartan subalgebras

For any f.d. Lie alg. $\mathfrak{g}/\mathbb{C}$: $\ell = \mathrm{rank}(\mathfrak{g})$ is the smallest j s.t.

$P_j \not\equiv 0$ where $P_n(x), \ldots, P_0(x)$ $(x \in \mathfrak{g})$ are defined by

$$\det(T \cdot \mathrm{Id}_{\mathfrak{g}} - \mathrm{ad}(x)) = \sum_{j=0}^{n} P_j(x) \cdot T^j$$

Alternately, $\forall y \in \mathfrak{g}$, $\mathfrak{g}_0(y) := \bigcup_{m \geq 0} \mathrm{Ker}(\mathrm{ad}(y)^m)$, then

$\dim \mathfrak{g}_0(y) \geq \ell$ and $\{x : P_\ell(x) \neq 0\} \longleftrightarrow \{x : \dim \mathfrak{g}_0(x) = \ell\}$

$\mathfrak{h} = \mathfrak{g}_0(x) \subset \mathfrak{g}$ is a nilpotent subalgebra s.t. $[y, \mathfrak{h}] \subset \mathfrak{h}$ $\Rightarrow y \in \mathfrak{h}$.

(here x is s.t. $P_\ell(x) \neq 0$) (Cartan Subalg.)

Prop (11.1) $\Rightarrow$ $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha$ $\quad \mathfrak{g}_\alpha := \{y \in \mathfrak{g} : (\mathrm{ad}(h) - \alpha(h))^N \cdot y = 0$ for $N \gg 0\}$

For $\mathfrak{g}$ semisimple, we proved (see (11.6) of Lecture 11, page 7)

- $\mathfrak{h}$ is abelian, $\mathrm{ad}(h) \circlearrowright \mathfrak{g}$ is semisimple operator $\forall h \in \mathfrak{h}$
- $K|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate

Let $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\}$

- R spans $\mathfrak{h}^*$ (i.e. $\alpha(h) = 0 \ \forall \alpha \in R \Rightarrow h = 0$)
- $\forall \alpha \in R$, $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is non-degenerate $\left(K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_\beta} \equiv 0 \text{ if } \alpha + \beta \neq 0\right)$

(12.1) Since $K|_{\mathfrak{h}\times\mathfrak{h}} : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}$ is non-degenerate, we have an iso.

$$\begin{array}{ccc} \mathfrak{h}^* & \longrightarrow & \mathfrak{h} \\ \cup\!\!\!\! \mid & & \cup\!\!\!\! \mid \\ \gamma & \longmapsto & t_\gamma \end{array}$$

where $K(t_\gamma, h) = \gamma(h)\ \forall h \in \mathfrak{h}$.

We also write $(\gamma, \gamma') = K(t_\gamma, t_{\gamma'})$.

In particular, $\forall \alpha \in R$, $t_\alpha \in \mathfrak{h}$ is a non-zero vector

Lemma: (i) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}\cdot t_\alpha \quad \forall \alpha \in R$

(ii) $\alpha(t_\alpha) \neq 0$

Proof: Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$. Then $K([x,y],h) = -K(y,[x,h])$
$= \alpha(h)\, K(y,x) = \alpha(h)\, K(x,y) \quad \forall h \in \mathfrak{h}$

$\Rightarrow [x,y] = K(x,y)\cdot t_\alpha \quad - \quad (*)$

hence (i). Now pick $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ s.t. $K(x_\alpha, y_\alpha) = 1$ (by non-deg. of $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$). Thus, by $(*)$ $[x_\alpha, y_\alpha] = t_\alpha$.

If $\alpha(t_\alpha) = 0$, $\mathfrak{k} = \mathbb{C}x_\alpha + \mathbb{C}y_\alpha + \mathbb{C}t_\alpha$ is a nilpotent Lie alg. Again Prop (11.1) applied to $\mathfrak{k} \circlearrowright_{ad} \mathfrak{g} \Rightarrow$ eigenvalues of $[\mathfrak{k},\mathfrak{k}]$ on $\mathfrak{g}$ are all zero. But $t_\alpha \in [\mathfrak{k},\mathfrak{k}]$ and its eigenvalues on $\mathfrak{g}$ are $\beta(t_\alpha)$ $(\beta \in R)$. So $\beta(t_\alpha) = 0\ \forall \beta \in R$, hence $t_\alpha = 0$, which is a contradiction. $\square$

(12.2) Define $h_\alpha = \dfrac{2t_\alpha}{\alpha(t_\alpha)}$ so that $\alpha(h_\alpha) = 2$.

Pick $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ so that $K(e_\alpha, f_\alpha) = \dfrac{2}{\alpha(t_\alpha)}$

and hence by $(*)$ $\quad [e_\alpha, f_\alpha] = h_\alpha$. Also $[h_\alpha, e_\alpha] = \alpha(h_\alpha)\, e_\alpha = 2e_\alpha$

and $[h_\alpha, f_\alpha] = -2f_\alpha$.

That is, we have
$$\begin{array}{ccc} sl_2 & \longrightarrow & \mathfrak{g} \circlearrowright_{ad} \mathfrak{g}. \\ e/f/h & \longmapsto & e_\alpha / f_\alpha / h_\alpha \end{array}$$

To simplify notation we just write $sl_2^{(\alpha)} \circlearrowright_{ad} \mathfrak{g}$.

(12.3) Lemma. (1) $\dim \mathfrak{g}_\alpha = 1 \quad \forall \alpha \in R$

(2) $\forall \alpha, \beta \in R$, define $a_{\alpha\beta} = \beta(h_\alpha) = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}$. Then

$$a_{\alpha\beta} \in \mathbb{Z}$$

and $\quad \beta - a_{\alpha\beta}\cdot\alpha \in R$

(3) $\mathfrak{g}_\alpha \neq 0$ & $\mathfrak{g}_{c\alpha} \neq 0 \Rightarrow c = \pm 1$.

[All from sl_2 - repn. theory - if $V \circlearrowleft sl_2$ is a f.d. repn and $v \in V$ is a non-zero vector s.t. $\begin{array}{l} hv = \lambda v \\ ev = 0 \end{array}$ then $\lambda \in \mathbb{Z}_{\geq 0}$.

Moreover if $V = \bigoplus_{j=0}^{\lambda} V_{\lambda - 2j}$ and $\dim V_{\lambda-2j} = 1$ then V is irr.]

Proof. (1) Assume $\dim \mathfrak{g}_\alpha$ ($= \dim \mathfrak{g}_{-\alpha}$ by non-deg of K) ≥ 2.

Then we can find $y \in \mathfrak{g}_{-\alpha}$ s.t. $K(e_\alpha, y) = 0$. But then for $\mathfrak{sl}_2^{(\alpha)} \circlearrowright \mathfrak{g}$, $\left.\begin{array}{l} h_\alpha \cdot Y = [h_\alpha, Y] = -2Y \\ e_\alpha \cdot Y = K(e_\alpha, Y). t_\alpha = 0 \end{array}\right\}$ contradiction.

(2) Let $\beta \in R$ and consider $\mathfrak{a} = \bigoplus_{\substack{t \in \mathbb{Z}: \\ \beta + t\alpha \in R}} \mathfrak{g}_{\beta + t\alpha} \circlearrowleft \mathfrak{sl}_2^{(\alpha)}$.

Then $\mathfrak{a}$ is an irreducible repn. of $\mathfrak{sl}_2$. h_α eigenvalue on $\mathfrak{g}_{\beta+t\alpha}$ is

$$\beta(h_\alpha) + 2t \in \mathbb{Z} \Rightarrow a_{\alpha\beta} \in \mathbb{Z}.$$

Weights must form a string

$$\beta + t\alpha, \quad \beta + (t-1)\alpha, \quad \dots$$

h_α eigenvalue $a_{\alpha\beta} + 2t$, $\quad \dots \quad$ last one must be $-a_{\alpha\beta} - 2t$

$\Rightarrow -a_{\alpha\beta}$ is in the list, corr to $(\beta - a_{\alpha\beta}\alpha)(h_\alpha)$. Hence

$\beta - a_{\alpha\beta}\alpha \in R$. (Recall: notation $s_\alpha(\beta) = \beta - a_{\alpha\beta}\alpha$;

s_α acts on $\mathfrak{h}^*$ via

$s_\alpha(\gamma) = \gamma - \gamma(h_\alpha)\alpha$ is reflection in $\{\xi : \xi(h_\alpha) = 0\}$ hyperplane)

(3) Let $\beta = c\alpha \in R$. Then $a_{\alpha\beta} = 2c \in \mathbb{Z}$

$$a_{\beta\alpha} = \frac{2}{c} \in \mathbb{Z}$$

$\Rightarrow c = \pm 1$ or ± 2. Assume $c = 2$. Then $\mathfrak{a}$ (from (2)) is

$\mathfrak{g}_{2\alpha} + \mathfrak{g}_\alpha + \mathbb{C}h_\alpha + \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$ irr repn. But $e_\alpha \cdot e_\alpha = 0$

is in weight space $\mathfrak{g}_\alpha$. Contradiction.

(12.4) Equivalent (to (10.4) of Lecture 10 page 4) defn. of root system. Let E be a f.d $\mathbb{R}$-vector space. A root system $R \subset E^* \setminus \{0\}$ is a finite set s.t.

(i) R spans E^*. $\alpha, c\alpha \in R \Rightarrow c = \pm 1$.

(ii) $\forall\, \alpha \in R$, $\exists!\, h_\alpha \in E$ (given) s.t. $\alpha(h_\alpha) = 2$. Define

$$s_\alpha(\xi) = \xi - \xi(h_\alpha)\alpha \;:\; E^* \longrightarrow E^*$$

(iii) $\forall\, \alpha, \beta \in R$, $\quad \beta(h_\alpha) \in \mathbb{Z}$ and $s_\alpha(\beta) \in R$.

From Lemma (12.3) $\left(\begin{array}{l} \text{take } E = \mathbb{R}\text{- span of } h_\alpha\text{'s} \subset \mathfrak{h} \\ E^* = \text{"} \quad \text{"} \quad \text{"} \;\alpha\text{'s} \subset \mathfrak{h}^* \end{array}\right)$

$\{\alpha \in \mathfrak{h}^* \setminus 0 : \mathfrak{g}_\alpha \neq 0\}$ is a root system.

(12.5) We will see later how to classify root systems in terms of Dynkin diagrams.

Idea : consider $E^\circ = E \setminus \bigcup_{\alpha \in R} \mathrm{Ker}(\alpha)$ - disconnected space

Pick $C \subset E^\circ$ a connected component (called fundamental chamber)

$\mathcal{B} = \{\alpha_1, \ldots, \alpha_\ell\}$ - walls of C

i.e. $\begin{cases} \mathrm{Ker}(\alpha_j) \cap \overline{C} \neq \phi \\ \alpha_j(x) > 0 \quad \forall\, j = 1, \ldots, \ell\,;\; x \in C \end{cases}$

$\mathcal{B} \subset R$ is a base of R (see page 6 of Lecture 10)

Now I will explain how to go from a Cartan Matrix to semisimple Lie algebra

(12.6) Input $\rightsquigarrow$ Output

$A = (a_{ij})_{1 \le i,j \le \ell}$

$a_{ii} = 2$

$a_{ij} \in \{0, -1, -2, -3\}$

$\det(A) > 0$

$\mathfrak{g}(A)$ · semisimple Lie alg.

Generators:

$h_i, e_i, f_i \quad (1 \le i \le \ell)$ [Chevalley generators]

Rel^{n}s: $[h_i, h_j] = 0$

$[h_i, e_j] = a_{ij} e_j \quad [h_i, f_j] = -a_{ij} f_j$

$[e_i, f_j] = \delta_{ij}$

+ Serre Relation $\forall\ i \neq j$

$$\mathrm{ad}(e_i)^{1-a_{ij}} \cdot e_j = 0 = (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j$$

$\mathfrak{h}$ = Span of $\{h_1, \dots, h_\ell\}$ ← Cartan Subalgebra

$\mathfrak{h}^* \ni \alpha_i$ defined by $\alpha_i(h_j) = a_{ji}$ [Simple Roots]

We get explicit formulae for $s_i = s_{\alpha_i}$ [Simple Reflections]

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$$

$W = \langle s_i : 1 \le i \le \ell \rangle$ [Weyl Group]

$R = \bigcup_{1 \le j \le \ell} W\alpha_j$ (Keep applying s_i's to α_j's)

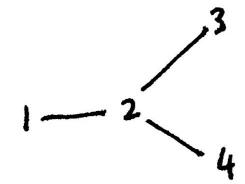
$\dim \mathfrak{g} = |R| + \ell$ (one basis vector for each $\alpha \in R$).
$\ell = \dim \mathfrak{h}$.

We can inductively construct a non-zero vector in any $\mathfrak{g}_\beta$ as

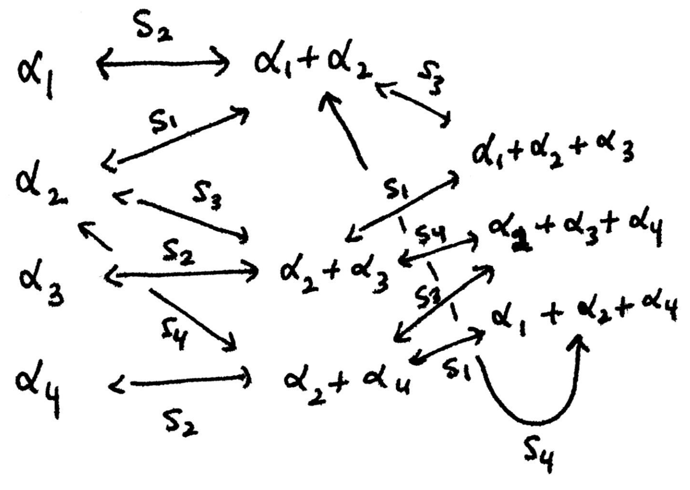
if $\beta > 0$ (i.e. +ve (or non-neg.) combination of α_i's)

$\beta, \beta + \alpha_i \in R \Rightarrow [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_\beta] = \mathfrak{g}_{\beta+\alpha_i}$. All the commutation relations can be worked out from Jacobi identity and relations given above.

(12.7) Example of D_4 :

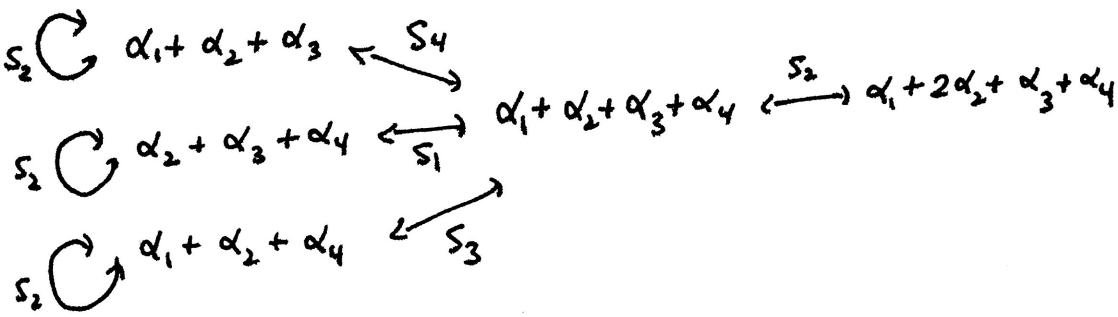


$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$



$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$

$(s_i(\alpha_i) = -\alpha_i)$



12 +ve roots   24 total $\dim \mathfrak{g}(D_4) = 24 + 4 = 28$

rank

$$\mathfrak{g}_{\alpha_1+\alpha_2} = \mathbb{C}\cdot[e_1, e_2] \quad : \text{span of } e_{\alpha_1+\alpha_2} := [e_1, e_2]$$

$$\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} = \text{Span of } [e_3\,[e_1\, e_2]] \quad (\text{note } [e_1, e_3] = 0 \text{ by Serre rel}^{\text{n}}\text{s}).$$

and so on. To figure out (all) the structure constants.

$$\begin{tikzcd}
 & \mathfrak{g}_{\alpha_1+\alpha_2} \arrow[dl, "\mathrm{ad}(f_1)"'] \arrow[dr, "\mathrm{ad}(f_2)"] & \\
\mathfrak{g}_{\alpha_2} & & \mathfrak{g}_{\alpha_1}
\end{tikzcd}$$

e.g. $$[f_1, [e_1, e_2]] = [[f_1\, e_1], e_2] + \overset{0 \text{ since } [e_i, f_j] = \delta_{ij} h_i}{\cancel{[e_1, [f_1, e_2]]}}$$

$$= -[h_1, e_2] = e_2 \quad (\text{as } a_{12} = -1).$$