Lecture 13

(13.0) Recall: $\mathfrak{g}$ is semisimple Lie alg. $/\mathbb{C}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. We get $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ ($R \subset \mathfrak{h}^* \setminus \{0\}$ consists of α s.t. $\mathfrak{g}_\alpha \neq \{0\}$)

Let $\Delta = \{\alpha_i : i \in I\} \subset R$ be a base of R (i.e. Δ is a basis of $\mathfrak{h}^*$ and $R = \underbrace{R_+}_{\text{positive roots}} \sqcup \underbrace{R_-}_{-R_+}$ where $R_\pm = R \cap \left(\sum_{i \in I} (\pm \mathbb{N}) \alpha_i\right)$.

Note: Last time we defined $h_\alpha \in \mathfrak{h}$ ($\forall \alpha \in R$) s.t. $\begin{cases} \alpha(h_\alpha) = 2 \\ \beta(h_\alpha) \in \mathbb{Z} \ \forall \beta \in R \end{cases}$

and showed that $R \subset$ Real span of α's $=: E^*$ is a finite root system.
(E = real span of h_α's)

$\forall h, h' \in \mathfrak{h}$ $\quad K(h, h') = \sum_{\alpha \in R} \alpha(h)\alpha(h')$ [(ii) of (11.6) Lecture 11, page 7]

$\Rightarrow$ K defines (real) positive definite form on $E = \sum_{\alpha \in R} \mathbb{R} h_\alpha$

Also, from Lemma (12.3) of Lecture 12, page 4, if $\alpha, \beta, \alpha+\beta \in R$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(13.1) I just want to remark that Cartan's criterion for semisimplicity works over $\mathbb{R}$. We only used Solvability Criterion to prove it (see (8.9) of Lecture 8, page 8). For solvability:

$$\mathfrak{a} \text{ is solvable} \iff K(\mathfrak{a}, [\mathfrak{a}, \mathfrak{a}]) = 0$$

Both sides are stable under extension of scalars from $\mathbb{R}$ to $\mathbb{C}$.

(13.2) Some notational conventions

$\mathfrak{k}$: Lie algebra over $\mathbb{R}$ $\qquad$ $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ Lie algebra over $\mathbb{C}$

extension of scalars

$\mathfrak{k} \subset \mathfrak{k}_{\mathbb{C}}$ is (real) Lie subalgebra and Killing form restricts as: $K_{\mathfrak{k}}(x,y) = K_{\mathfrak{k}_{\mathbb{C}}}(x,y) \quad \forall\, x,y \in \mathfrak{k}$.

$\mathfrak{g}$: Lie algebra over $\mathbb{C}$ $\qquad$ $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}$ considered as Lie alg. (of twice the dim) over $\mathbb{R}$

(restriction of scalars)

In this case $K_{\mathfrak{g}^{\mathbb{R}}}(x,y) = 2\,\mathrm{Re}\left(K_{\mathfrak{g}}(x,y)\right) \quad \forall\, x,y \in \mathfrak{g}$.

(13.3) Definition. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. A <u>real form</u> of $\mathfrak{g}$, is a real subalgebra $\mathfrak{a} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C} \quad (= \mathfrak{a}_{\mathbb{C}}).$$

A real form $\mathfrak{a}$ of $\mathfrak{g}$ is said to be <u>compact real form</u> if in addition $K_{\mathfrak{g}}\big|_{\mathfrak{a}\times\mathfrak{a}}$ ($= K_{\mathfrak{a}}$ - see (13.2)) is negative definite. That is $K(x,x) < 0 \quad \forall\, x \in \mathfrak{a}$.

Note: if $\mathfrak{a} \subset \mathfrak{g}$ is a real form, Killing form on $\mathfrak{a}$ is same as Killing form on $\mathfrak{g}$, hence $\mathfrak{a}$ is again semisimple.

(13.4) Example. $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ has a basis $\{h, e, f\}$ s.t. the structure constants are $\{\pm 2, 0\}$ real. so $\mathfrak{sl}_2(\mathbb{R})$ = real span of $\{h, e, f\}$ is a real form of $\mathfrak{g}$; but not a compact real form.

Let us normalize Killing form so that
$$(e, f) = 1$$
$$(h, h) = 2$$

Then $\mathbb{R}$-span of $\underbrace{ih}_{=\sigma^0}$, $\underbrace{e-f}_{=\sigma^+}$, $\underbrace{i(e+f)}_{=\sigma^-}$ is a compact real form:

$(\sigma^0, \sigma^0) = -2 \qquad (\sigma^+, \sigma^+) = -2 \qquad (\sigma^-, \sigma^-) = -2$

$(\sigma^0, \sigma^\pm) = 0 \qquad (\sigma^+, \sigma^-) = 0$

$X \in \mathbb{R}$-span of $\sigma^0, \sigma^\pm \iff \overline{X^T} = -X$, $\mathrm{Tr}(X) = 0$

i.e. $\mathbb{R}\{\sigma^0, \sigma^\pm\} = \mathfrak{su}_2$ = Lie algebra of $SU_2 = \{x \in M_{2\times 2}(\mathbb{C}) : \overline{x}^t = x^{-1},\ \det(x) = 1\}$

compact Lie group

(13.5) Theorem. Every semisimple Lie algebra $\mathfrak{g}$ admits a compact real form. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are two compact real forms then $\exists\ \eta \in \mathrm{Aut}_{\text{Lie Alg.}}(\mathfrak{g})$ s.t. $\eta(\mathfrak{a}_1) = \mathfrak{a}_2$.

Later we will prove that if $\mathfrak{a} = \mathrm{Lie}(G)$ is real semisimple, and G is compact, then Killing form on $\mathfrak{a}$ is negative definite.

Combined with the classification of complex semisimple Lie algebra, Thm (13.5) will prove the classification of compact (real) semisimple Lie groups.

(13.6) Proof of Thm (13.5) existence. Let $\{h_i, e_i, f_i\}_{i\in I}$ be Chevalley gen. of $\mathfrak{g}$. Let $\varphi \in \mathrm{Aut}(\mathfrak{g})$ be defined by $\varphi(h_i) = -h_i$

$\varphi(e_i) = f_i$ and $\varphi(f_i) = e_i$

It is trivial to check that φ preserves all the defining relations of $\mathfrak{g}$

e.g. $[\varphi(e_i), \varphi(f_i)] = [f_i, e_i] = -h_i = \varphi(h_i)$

$$[\varphi(h_i), \varphi(e_j)] = -[h_i, f_j] = a_{ij} f_j = a_{ij}\varphi(e_j)$$

Choose $x_\alpha \in \mathfrak{g}_\alpha$ $\forall \alpha \in R$ s.t. $K(x_\alpha, x_{-\alpha}) = 1$. That is, $[x_\alpha, x_{-\alpha}] = t_\alpha$ (see formula (*) on page 2 of Lecture 12). Let $c_\alpha \in \mathbb{C}^\times$ be given by

$$\varphi(x_\alpha) = c_{-\alpha} x_{-\alpha}$$

As $K(\cdot,\cdot)$ is invariant under $\mathrm{Aut}(\mathfrak{g})$ [$\forall\ \sigma \in \mathrm{Aut}(\mathfrak{g})$, $x \in \mathfrak{g}$, we have

$$\mathrm{ad}(\sigma(x))\cdot y = [\sigma(x), y] = \sigma[x, \sigma^{-1}(y)] = \sigma\,\mathrm{ad}(x)\,\sigma^{-1}.$$ Hence

$$K(\sigma x, \sigma y) = \mathrm{Tr}(\mathrm{ad}(\sigma x)\,\mathrm{ad}(\sigma y)) = \mathrm{Tr}(\sigma\,\mathrm{ad}(x)\,\mathrm{ad}(y)\,\sigma^{-1}) = \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$$
$$= K(x,y) \quad \forall\ x, y \in \mathfrak{g}\ ]$$

We get $\quad c_\alpha c_{-\alpha} = 1$

Now pick $a_\alpha, a_{-\alpha} \in \mathbb{C}^\times$ s.t.
$$a_\alpha a_{-\alpha} = 1$$
$$a_\alpha^2 = -c_\alpha$$

and define $X_\alpha := a_\alpha x_\alpha \in \mathfrak{g}_\alpha$.

(1) $[X_\alpha, X_{-\alpha}] = a_\alpha a_{-\alpha} [x_\alpha, x_{-\alpha}] = t_\alpha$.

(2) $\varphi(X_\alpha) = a_\alpha c_{-\alpha} x_{-\alpha} = \dfrac{c_{-\alpha}}{a_{-\alpha}} x_{-\alpha} = -a_{-\alpha} x_{-\alpha} = -X_{-\alpha}$

Define $N_{\alpha,\beta}$ by $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ if $\alpha+\beta \in R$

(3) $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$. Proof. $-N_{\alpha,\beta} X_{-\alpha-\beta} = \varphi(N_{\alpha,\beta} X_{\alpha+\beta})$

$= \varphi([X_\alpha, X_\beta]) = [\varphi(X_\alpha), \varphi(X_\beta)] = [X_{-\alpha}, X_{-\beta}] = N_{-\alpha,-\beta} X_{-\alpha-\beta}$ $\square$

<u>Lemma</u>. (a) $N_{\alpha,\beta} = -N_{\beta,\alpha}$ (this is clear)

(b) if $\alpha, \beta, \gamma \in R$ s.t. $\alpha+\beta+\gamma = 0$ then

$$N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$$

(c) Let $\alpha, \beta \in R$ and (according to Proof of Lemma (12.3) (2) of Lecture 12, page 4) $\{\beta - p\alpha, \beta-(p-1)\alpha, \ldots, \beta+q\alpha\} \subset R$ where $\beta-(p+1)\alpha$, $\beta+(q+1)\alpha \notin R$. Then

$$N_{\alpha,\beta}^2 = \frac{1}{2} q (p+1) |\alpha|^2$$

(d) $|\alpha|^2 = \alpha(t_\alpha) = K(t_\alpha, t_\alpha) \in \mathbb{Q}$

Hence $N_{\alpha,\beta}$ are all real. and $(\alpha,\beta) = \beta(t_\alpha) = \alpha(t_\beta) \in \mathbb{Q}$ $\forall \alpha, \beta \in R$

Assuming this Lemma, let $\mathfrak{a} = \mathbb{R}$-span of

$$\{ it_\alpha ,\ X_\alpha - X_{-\alpha} ,\ i(X_\alpha + X_{-\alpha}) \}_{\alpha \in R}$$

Notation: $\sigma_\alpha^0 = it_\alpha \quad \sigma_\alpha^+ = X_\alpha - X_{-\alpha} \quad \sigma_\alpha^- = i(X_\alpha + X_{-\alpha})$.

Easy computations: $\quad [\sigma_\alpha^0, \sigma_\beta^0] = 0$

$$[\sigma_\alpha^0, \sigma_\beta^\pm] = \pm \beta(t_\alpha)\, \sigma_\beta^\mp$$

$$\left.\begin{aligned} [\sigma_\alpha^+, \sigma_\beta^+] &= N_{\alpha,\beta}\, \sigma_{\alpha+\beta}^+ - N_{-\alpha,\beta}\, \sigma_{\beta-\alpha}^+ \\ [\sigma_\alpha^+, \sigma_\beta^-] &= N_{\alpha,\beta}\, \sigma_{\alpha+\beta}^- - N_{-\alpha,\beta}\, \sigma_{\beta-\alpha}^- \\ [\sigma_\alpha^-, \sigma_\beta^-] &= -N_{\alpha,\beta}\, \sigma_{\alpha+\beta}^+ - N_{-\alpha,\beta}\, \sigma_{\beta-\alpha}^+ \end{aligned}\right\} \quad \beta \neq \pm\alpha$$

$$[\sigma_\alpha^+, \sigma_\alpha^-] = 2\sigma_\alpha^0 \qquad \Rightarrow \mathfrak{a} \text{ is a real form of } \mathfrak{g}.$$

Finally, K is negative definite on $\sum_{\alpha \in R} \mathbb{R}\, i t_\alpha$.

$K(\sigma_\alpha^\pm, \sigma_\alpha^\pm) = -2$ all others 0

$\Rightarrow \mathfrak{a}$ is a compact real form

□

$\{t_\beta, X_\alpha\}_{\substack{\alpha \in R \\ \beta \in \Delta = \{\alpha_1, \dots, \alpha_\ell\}}}$ is often refered to as <u>Weyl basis of $\mathfrak{g}$</u>.

(13.7) Proof of Lemma (13.6) on page 5: Fix $\alpha \in R$ and use (2) of Lemma (12.3) page 4 of Lecture 12, to define $\forall \beta \in R$, $p_\beta, q_\beta \in \mathbb{Z}_{\geq 0}$ s.t. $\{\beta - p_\beta \alpha, \ldots, \beta + q_\beta \alpha\} \subset R$

"α-string through β".

Thus $\beta(h_\alpha) + 2q_\beta = -\beta(h_\alpha) + 2p_\beta$

$$\Rightarrow \quad p_\beta - q_\beta = \beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$$

Now
$$(\alpha,\alpha) = K(t_\alpha, t_\alpha) = \sum_{\beta \in R} \beta(t_\alpha)^2 = \frac{(\alpha,\alpha)^2}{4} \sum_{\beta \in R} (p_\beta - q_\beta)^2$$

Since $(\alpha,\alpha) = \alpha(t_\alpha) \neq 0$ (Lemma (12.1) of Lecture 12 page 2), we get
$$(\alpha,\alpha) = \frac{4}{\sum_{\beta \in R} (p_\beta - q_\beta)^2} \in \mathbb{Q}.$$
This proves (d).

(b) follows from Jacobi id.: $[[X_\alpha X_\beta] X_\gamma] + [[X_\beta, X_\gamma], X_\alpha] + [[X_\gamma, X_\alpha] X_\beta] = 0$

$$\Rightarrow \quad N_{\alpha,\beta}\, t_\gamma + N_{\beta,\gamma}\, t_\alpha + N_{\gamma,\alpha}\, t_\beta = 0$$

$t_\gamma = t_{-\alpha-\beta} = -t_\alpha - t_\beta$ and linear independence of α, β implies

$$N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$$

Recall repn. th. of sl_2 : irr. repn. V of dim $\mu+1$ (h.w. $=\mu$)
highest weight

basis $\{v_j = f^j v_0\}_{0\le j\le\mu}$

$$e v_j = j(\mu-j+1)\, v_{j-1}$$

We take $V = \sum_{-p\le t\le q} \mathfrak{g}_{\beta+t\alpha}$ $\qquad \mu = \beta+q\alpha\,(h_\alpha) = \beta(h_\alpha)+2q$

$\mathfrak{g}_\beta$ corr to $\mathbb{C}\cdot v_q$ in the notation above

We get $\quad [f_\alpha, [e_\alpha, X_\beta]] = q\,(\beta(h_\alpha)+q+1)\, X_\beta$

$$= q\,(p+1)\, X_\beta$$

Recall $\quad e_\alpha = X_\alpha$

$f_\alpha = \dfrac{2}{\alpha(t_\alpha)} X_{-\alpha}$ $\qquad$ so that $[e_\alpha, f_\alpha] = h_\alpha = \dfrac{2}{\alpha(t_\alpha)}\cdot t_\alpha$

$$\Rightarrow \quad [X_{-\alpha}\ [X_\alpha, X_\beta]] = q\,(p+1)\,\frac{\alpha(t_\alpha)}{2}\, X_\beta$$

$$\Rightarrow \quad N_{-\alpha,\alpha+\beta}\, N_{\alpha,\beta} = q\,(p+1)\,\frac{|\alpha|^2}{2}.$$ Now we are done

since $\quad N_{-\alpha,\alpha+\beta} = N_{-\beta,-\alpha}$ $\quad$ by (b)

$= -N_{-\alpha,-\beta}$ $\quad$ by (a)

$= N_{\alpha,\beta}$ $\quad$ by (3) of page 5.