Lecture 14

(14.0) Recall that for a complex semisimple Lie algebra $\mathfrak{g}$, we defined a compact real form: a real Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ s.t. $\mathfrak{g} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$

and $K_{\mathfrak{g}}\Big|_{\mathfrak{a}} : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathbb{C}$ (necessarily real-valued)

is negative-definite.

We proved the existence of compact real forms by constructing a Weyl basis of $\mathfrak{g}$.

Today we will prove the uniqueness of compact real forms.

(14.1) Let $\mathfrak{a} \subset \mathfrak{g}$ be a compact real form. $\mathfrak{a}$ defines an <u>involution</u> on $\mathfrak{g}$ as

$$\begin{aligned} \mathfrak{g} = \mathfrak{a} \oplus i\mathfrak{a} &\xrightarrow{\tau} \mathfrak{g} = \mathfrak{a} \oplus i\mathfrak{a} \\ (X + iY) &\longmapsto X - iY \end{aligned}$$

- τ is $\mathbb{R}$-linear automorphism of Lie algebra $\mathfrak{g}$ (i.e. let $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}$ considered as a real space, then $\tau \in \mathrm{Aut}_{\text{Lie Alg}}(\mathfrak{g}^{\mathbb{R}})$.)

- $\tau^2 = \mathrm{Id}_{\mathfrak{g}}$

- $\forall\, x \in \mathfrak{g}$, $\quad K(x, \tau(x)) \in \mathbb{R}_{\leq 0} \quad (= 0 \iff x = 0)$

Pf. Write $X = X_{re} + i X_{im} \quad (X_{re}, X_{im} \in \mathfrak{a})$

Then $K(x, \tau(x)) = K(X_{re}, X_{re}) + K(X_{im}, X_{im}) \leq 0$

$= 0 \iff X_{re} = X_{im} = 0$ (since $\mathfrak{a}$ is a cpct real form $K(X_{re}, X_{re}), K(X_{im}, X_{im}) \leq 0$)

Remark. τ does not preserve $K_{\mathfrak{g}}(\cdot,\cdot)$ since it is not a $\mathbb{C}$-linear automorphism. However it does preserve $K_{\mathfrak{g}^{\mathbb{R}}}$.

Note $K_{\mathfrak{g}^{\mathbb{R}}}(x,y) = 2\,\mathrm{Re}\left(K_{\mathfrak{g}}(x,y)\right)$. $\forall\, x,y \in \mathfrak{g}$

In particular, $K_{\mathfrak{g}^{\mathbb{R}}}$ is non-degenerate (hence $\mathfrak{g}^{\mathbb{R}}$ remains semisimple).

(14.2) From now on, let k be an arbitrary real semisimple Lie algebra.

Defn. A <u>Cartan involution</u> on k is a Lie alg. automorphism $\tau : k \longrightarrow k$ s.t. $\tau^2 = Id_k$ and $\forall\, x,y \in k$

$(x,y)_\tau := -K(x,\tau(y))$ is a positive-definite bilinear form.

Thm. Let σ, τ be two Cartan involutions on k. Then $\exists\, \rho \in \underset{\text{Lie Alg}}{\mathrm{Aut}}(k)$ (in fact in the connected component of id) s.t. $\sigma = \rho \tau \rho^{-1}$.

Cor (of Thm and its proof). If $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{g}$ are two cpct real forms, then $\exists\, \rho \in \mathrm{Aut}(\mathfrak{g})$ s.t. $\rho(\mathfrak{a}_2) = \mathfrak{a}_1$.

Pf. Let σ, τ = involutions assoc. to $\mathfrak{a}_1$ and $\mathfrak{a}_2$ resp. Take $k = \mathfrak{g}^{\mathbb{R}}$ in the theorem. As we will see in the proof

$\rho \in \mathrm{Aut}(\mathfrak{g})$ (over $\mathbb{C}$). Since $\mathfrak{a}_1 = \mathfrak{g}^\sigma$ $(= \{x : \sigma(x) = x\})$

and $\mathfrak{a}_2 = \mathfrak{g}^\tau$ we get

$$\mathfrak{a}_1 = \mathfrak{g}^\sigma = \mathfrak{g}^{\rho\tau\rho^{-1}} = \rho(\mathfrak{g}^\tau) = \rho(\mathfrak{a}_2) \text{ as required} \quad \square$$

(14.3) Lemma. σ, τ, k as in Thm (14.2). If $\sigma\tau = \tau\sigma$ then $\sigma = \tau$.

Proof. Since $\sigma^2 = \tau^2 = 1$ and they commute, we can diagonalize them simultaneously. We claim that $\sigma(x) = x \Leftrightarrow \tau(x) = x$.
(eigenvalues ± 1)

Because if there is $x \in k$ s.t. $\sigma(x) = x$ and $\tau(x) = -x$, we get

$$\left.\begin{aligned} (x,x)_\sigma &= -K(x,x) \geq 0 \\ (x,x)_\tau &= +K(x,x) \geq 0 \end{aligned}\right\} \Rightarrow x = 0$$

(positive-definiteness of $(\cdot,\cdot)_\sigma$ & $(\cdot,\cdot)_\tau$) $\square$

(14.4) Proof of Thm (14.2). It is enough to construct ρ s.t. σ and $\rho\tau\rho^{-1}$ commute

Let $\omega = \sigma\tau : k \longrightarrow k$.

Claim. ω^2 is symmetric, positive-definite w.r.t. $(\cdot,\cdot)_\sigma$.

i.e. $(x, \omega^2 y)_\sigma = (\omega^2 x, y)_\sigma$

and $(x, \omega^2 x)_\sigma \geq 0$ $(= 0 \Leftrightarrow x = 0)$

Proof of the claim.

$$(X, \omega^2 Y)_\sigma = -K(X, \sigma\omega^2 Y) = -K(X, \tau\sigma\tau Y)$$
$$= -K(\tau\sigma\tau(X), Y) = -K(\sigma\tau\sigma\tau(X), \sigma(Y))$$
$$= (\omega^2(X), Y)_\sigma .$$

$$(X, \omega^2 X)_\sigma = -K(X, \sigma\omega^2 X) = -K(X, \tau\sigma\tau(X))$$
$$= -K(\tau(X), \sigma\tau(X)) = (\tau(X), \tau(X))_\sigma > 0$$

Thus $\varphi = \omega^2$ is diagonalizable with +ve (real) eigenvalues.

Explicitly, there is a basis of k $\{x_1, \dots, x_n\}$ s.t.

$\varphi(x_j) = \varphi_j x_j$ $(\varphi_j \in \mathbb{R}_{>0})$. is a Lie alg. auto. of k.

$\Longrightarrow$ $\varphi_t : x_j \longmapsto \varphi_j^t x_j$ is again a Lie alg. auto. of k $(\forall t \in \mathbb{R})$

Thus $t \longmapsto \varphi_t$ is a 1-parameter subgroup in $\mathrm{Aut}(k)$

(hence lies in the connected component of the identity).

~~Take~~ $\cancel{\rho =}$ Note: φ commutes with $\omega = \sigma\tau$ (clear)

and $\varphi\tau = \tau\varphi^{-1}$ (i.e. $\sigma\tau\sigma\tau\,\tau = \tau\,\tau\sigma\tau\sigma$ ✓)

$\Longrightarrow$ $\varphi_t \tau = \tau \varphi_t^{-1}$ $(\forall\ t \in \mathbb{R})$

Take $\rho = \varphi_{1/4}$. Then $\sigma(\rho\tau\rho^{-1}) \overset{?}{=} \rho\tau\rho^{-1}\sigma$

$$\Longleftrightarrow \sigma\tau\rho^{-2} \overset{?}{=} \rho^2\tau\sigma$$
$$\Longleftrightarrow \rho^{-2}\sigma\tau = \rho^2\tau\sigma$$

$\iff \quad \sigma\tau \overset{?}{=} \rho^4 \tau\sigma$. But $\rho^4 = \varphi = \sigma\tau\sigma\tau$ and the last eq^n is true.

(14.5) ρ is defined over $\mathbb{C}$.

Note that $\partial = \frac{d}{dt}\varphi_t \Big|_{t=0}$ is a derivation of $\mathfrak{k}$ (namely, the one that maps $x_j \longmapsto \ln(\varphi_j)\, x_j$). As $\mathfrak{k}$ is semisimple $\partial = \mathrm{ad}(x)$ for some $x \in \mathfrak{k}$. (*), and $\varphi_t = e^{t\,\mathrm{ad}(x)}$ $(\forall t \in \mathbb{R})$.

Now if $\mathfrak{k} = \mathfrak{g}^{\mathbb{R}}$, take the same $x \in \mathfrak{g}$ which defines $\varphi_t = e^{t\,\mathrm{ad}(x)} \in \mathrm{Aut}_{\text{Lie Alg}}(\mathfrak{g})$.

(*) we proved this using Weyl's complete reducibility (over $\mathbb{C}$).

<u>Proof</u> (for $\mathbb{R}$). Using Cartan's criterion. Since the Killing form is non-degenerate we can find x s.t. $K(x,y) = \mathrm{Tr}_{\mathfrak{k}}(\partial \circ \mathrm{ad}(y))$ $\forall y \in \mathfrak{k}$

$[\partial(y), z] = \partial([y,z]) - [y, \partial(z)]$

is same as: $\quad \mathrm{ad}(\partial(y)) = [\partial, \mathrm{ad}(y)]$. Now

$$K(\partial(y), z) = \mathrm{Tr}(\mathrm{ad}(\partial(y)) \circ \mathrm{ad}(z)) = \mathrm{Tr}([\partial, \mathrm{ad}(y)] \circ \mathrm{ad}\, z)$$

$$= \mathrm{Tr}(\partial \circ \mathrm{ad}\,[y,z]) = K(x, [y,z]) = K([x,y], z)$$

Non-deg. of $K \Rightarrow \partial(y) = [x,y] \quad \forall y \in \mathfrak{k}$. $\square$