Lecture 16

(16.0)
~~(14.0)~~ Aim: to define (real) compact simply-connected Lie groups of type B, C & D.

Type B_n - $Sp(n)$ as automorphisms of quaternion space

C_n $\quad Spin(2n+1)$ $\}$ using Clifford algebras

D_n $\quad Spin(2n)$

(16.1)
~~(14.1)~~ Quaternions: Q is a 4-dim'l real vector space with basis $\{e_0, e_1, e_2, e_3\}$ and multiplication given by: $e_0^2 = e_0$ and

$$e_0 e_i = e_i = e_i e_0 \quad ; \quad e_i^2 = -e_0 \qquad (i = 1, 2, 3)$$

$$e_i e_j = -e_j e_i \qquad e_1 e_2 = e_3 \, ; \; e_2 e_3 = e_1 \, ; \; e_3 e_1 = e_2$$

$$\left(\forall \, i, j \in \{1, 2, 3\} \atop (i \neq j) \right)$$

For $q = \sum_{\ell=0}^{3} a_\ell e_\ell \qquad \bar{q} := a_0 e_0 - \sum_{\ell=1}^{3} a_\ell e_\ell$ so that

$$q \bar{q} = \bar{q} q = \sum_{\ell=0}^{3} a_\ell^2 =: |q|^2 \qquad \left(\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1 \right)$$

$\Rightarrow$ Q is a division algebra with $q^{-1} = \dfrac{\bar{q}}{|q|^2} \quad \forall \, q \neq 0$

e_0 is the unit.

(16.2)
~~(14.2)~~ $\underline{Q \simeq \mathbb{C}^2}$: Identify e_1 with $i = \sqrt{-1}$

$\forall \, q = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$, we write

$$q = (a_0 e_0 + a_1 e_1) + (a_2 e_0 + a_3 e_1) e_2$$

$$= z_0 + w_0 \cdot j \qquad (z_0 = a_0 + i a_1 \, , \, w_0 = a_2 + i a_3 \in \mathbb{C})$$

Here $j = e_2$ satisfies $j^2 = -1$ $\quad i \cdot j = -j \cdot i$, or more generally $z \cdot j = j \cdot \bar{z}$ $\quad \forall z \in \mathbb{C}$. Thus multiplication in Q can be rewritten as : $\forall z_0, z_1, w_0, w_1 \in \mathbb{C}$

$$(z_0 + z_1 j)(w_0 + w_1 j) = (z_0 w_0 - z_1 \bar{w}_1) + (z_1 \bar{w}_0 + z_0 w_1) j$$

And conjugation $\overline{z_0 + z_1 j} = \bar{z}_0 - z_1 j$

(16.3) ~~(14.3)~~

$$Sp(n) := \left\{ \begin{array}{c} Q\text{-linear auto. } Q^n \xrightarrow{\sigma} Q^n \text{ s.t.} \\ (\sigma \underline{a}) \cdot (\sigma \underline{b}) = \underline{a} \cdot \underline{b} \end{array} \right\}$$

where $\underline{a} \cdot \underline{b} := \sum_{r=1}^{n} \bar{a}_r b_r \in Q$

Remark : Q^n is n-dim'l vector space over Q. Since Q is not commutative, in order to write linear maps as matrices with entries from Q, and elements of Q^n as column vectors we will have to consider Q^n as <u>right</u> Q-vector space (i.e. scalar multiplication is on the right

$$\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \cdot q = \begin{bmatrix} q_1 q \\ \vdots \\ q_n q \end{bmatrix}$$

~~(14.4)~~ (16.4) $Sp(1)$ = group of (square) length 1 quaternions

$$= \left\{ (a_0, a_1, a_2, a_3) \in \mathbb{R}^4 : \sum_{l=0}^{3} a_l^2 = 1 \right\} \simeq S^3$$

is compact and simply-connected.

Analogous to the case of $SU(n)$ - problem 3 of Homework 2
$SO(n)$ - see (6.1) of Lecture 6 page 2

we have $$Sp(n) \big/ Sp(n-1) \simeq S^{4n-1}$$

Using (5.4) of Lecture 5, page 6 : recall that if $H \overset{\text{closed}}{\subset} G$ (Lie gp.)
s.t. G/H is simply-connected then

$$\pi_1(H) \longrightarrow\!\!\!\!\to \pi_1(G)$$

(also Lemma (6.0) of Lecture 6 : H and G/H connected
$\Rightarrow$ G is connected)

We get $\{Sp(n)\}_{n \geq 1}$ are all compact and simply-connected.

(16.5) ~~(14.5)~~ An example of covering map

$$\begin{array}{c} Sp(1) \\ \downarrow \eta \\ SO(3) \end{array}$$

Consider the conjugation action : $q \in Sp(1) \rightsquigarrow \theta(q) : Q \to Q$

$$\theta(q) \cdot r = q\, r\, q^{-1}$$

Note : $\theta(q)$ maps pure quaternions $\left(\sum_{\ell=1}^{3} a_\ell e_\ell\right)$ to pure quaternions, since $r \in Q$ is pure $\Leftrightarrow$ $\bar{r} = -r$. Then

$$\overline{q r q^{-1}} = \overline{q^{-1}}\ \bar{r}\ \bar{q} = q(-r)\bar{q}^{-1} = -q r \bar{q}^{-1}.$$

Let $\eta(q)$ denote the 3×3 matrix of real numbers defined by the action of $\theta(q)$ on pure quaternions $\simeq \mathbb{R}^3$.

Now $\quad \theta(q)\cdot(r_1 r_2) = \theta(q)(r_1)\cdot\theta(q)(r_2) \quad \forall\, r_1, r_2 \in Q.$

If $r_1 = x_1 e_1 + x_2 e_2 + x_3 e_3$ $\qquad$ coeff of e_0 in $r_1 \cdot r_2 = -(x_1 y_1 + x_2 y_2 + x_3 y_3)$
$\quad r_2 = y_1 e_1 + y_2 e_2 + y_3 e_3$ $\qquad$ must be preserved under $\theta(q)$

i.e. $\eta(q)$ preserves the standard inner product on $\mathbb{R}^3$.

So : $\eta : Sp(1) \longrightarrow O(3)$. Now $Sp(1)$ is connected and $\eta(e_0) = \mathrm{Id}$ implies image of η lies in the connected component of $\mathrm{Id} \in O(3)$, i.e. $SO(3)$.

(16.6) ~~(14.6)~~ Lemma. $\eta : \cancel{SO(3)}\ Sp(1) \longrightarrow SO(3)$ is surjective

$\mathrm{Ker}(\eta) = \{\pm e_0\}$. η is a covering map

(Hence $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, and using the same argument as before. $\pi_1(SO(n))$ is at most of order 2 $\forall\, n \geq 4$)

$$SO(n)/SO(n-1) \simeq S^{n-1} \quad (n \geq 4)$$

Proof. $\mathrm{Ker}(\eta) = \{\pm e_0\}$: If $q \in Sp(1)$ is such that $\eta(q) = \mathrm{Id}$ then q commutes with e_1, e_2 & e_3. One can easily check that this means $q = a_0 e_0$, with $a_0^2 = 1 \Rightarrow q = \pm e_0$.

η is surjective: Easy check $\eta(\cos t\, e_0 + \sin t\, e_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{bmatrix}$

G_x = rotations around x-axis $\subset SO(3)$

G_y = " " y-axis $\subset SO(3)$

Thus we verify directly that (using Easy check) $G_x, G_y \subset \mathrm{Im}(\eta)$. But $SO(3)$ is generated by G_x and G_y (let $r \in SO(3)$ and $v = r(e_1)$ $\left(e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\ x\text{-axis}\right)$ using $g_1 \in G_x$ we can make $g_1 v$ lie in x-z plane, and then using $g_2 \in G_y$ we can make $g_2 g_1 v = e_1 = g_2 g_1 r(e_1)$. $\Rightarrow$ $g_2 g_1 r \in G_x$ and hence $r \in$ group gen. by G_x and G_y).

η is a covering map. This is immediate now since

$$\begin{tikzcd} Sp(1) \arrow[r, "2:1 \text{ covering map}"] \arrow[d, "\eta"'] & Sp(1)/\pm 1 \arrow[dl, "\bar\eta"] \\ SO(3) & \end{tikzcd}$$

$\bar\eta$ ← bijective continuous map between compact (Hausdorff) spaces is automatically a homeomorphism $\square$

(16.7) ~~(14.7)~~ $Sp(n) = U(2n) \cap Sp(n;\mathbb{C})$ via $\mathbb{Q}^n \simeq \mathbb{C}^n + \mathbb{C}^n \simeq \mathbb{C}^{2n}$

Here consider the standard symplectic form on $\mathbb{C}^{2n}$ - basis

$\{\varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_n\}$, $\quad (\varepsilon_i, \varepsilon_j) = 0 = (\eta_i, \eta_j) \quad \forall\, 1 \leq i, j \leq n$

$$(\varepsilon_i, \eta_j) = \delta_{ij} = -(\eta_j, \varepsilon_i)$$

and $Sp(n;\mathbb{C})$ are linear automorphisms of $\mathbb{C}^{2n}$ preserving this symplectic form.

Proof - Left as a homework exercise!