Lecture 17

(17.0)

~~(15.0)~~ Clifford algebra Cl_n over $\mathbb{R}$. Here $n \geq 1$.

Generators: $e_1, \dots, e_n$ (and e_0 = unit, i.e. $e_0^2 = e_0$, $e_0 e_i = e_i = e_i e_0$ $\forall i = 1, \dots, n$)

Relations: $e_i^2 = -e_0$ and $e_i e_j = -e_j e_i$ $\forall i \neq j \in \{1, \dots, n\}$

Thus $\dim Cl_n = 2^n$. We have a basis vector e_A for any subset $A \subset \{1, \dots, n\}$: $e_\phi := e_0$

if $i_1 < \dots < i_k$ are elements of A, then

$e_A := e_{i_1} \cdots e_{i_k}$

(17.1)

~~(15.1)~~ Lemma. (i) If n is even, the center of Cl_n is $\mathbb{R}e_0$. The only ideals of Cl_n are $\{0\}$ and Cl_n.

(ii) If n is odd, the center of Cl_n is $\mathbb{R}e_0 + \mathbb{R}f$, where $f = e_{\{1,\dots,n\}} = e_1 e_2 \cdots e_n$. Let j be such that $j^2 = (-1)^{n(n+1)/2}$ * and set $u = \frac{1}{2}(e_0 + jf)$ $v = \frac{1}{2}(e_0 - jf)$. Then the only ideals of Cl_n are $\{0\}$, $Cl_n \cdot u$, $Cl_n \cdot v$, Cl_n.

* if $n \equiv 1 \bmod 4$, we'll have to extend scalars and work over $\mathbb{C}$, ($j = \sqrt{-1}$).

Proof. $\forall h \in \{1, \dots, n\}$, define $C_h : Cl_n \to Cl_n$

$$C_h(x) = \frac{1}{2}\left(x - e_h x e_h\right)$$

Claim : for any $A \subset \{1, \dots, n\}$, we have

- if $|A|$ is even, $C_h(e_A) = \begin{cases} 0 & \text{if } h \in A \\ e_A & \text{if } h \notin A \end{cases}$

- if $|A|$ is odd, $C_h(e_A) = \begin{cases} e_A & \text{if } h \in A \\ 0 & \text{if } h \notin A \end{cases}$

Proof of the claim (for $|A|$ even). If $h \notin A$, we get

$$e_h e_A e_h = (-1)^{|A|} e_h^2 e_A = -e_A \Rightarrow C_h(e_A) = e_A .$$

If $h \in A$, say $A = i_1 < \dots < \underset{\overset{\uparrow}{= h}}{i_t} < \dots < \underset{\llcorner \text{ K is even}}{i_k}$. then

$$e_h e_A e_h = (-1)^{t-1+k-t} e_{i_1} \cdots e_{i_{t-1}} e_h e_h e_h e_{i_{t+1}} \cdots e_{i_k} \quad (\& \; e_h^2 = -e_0)$$

$$= (-1)(-1) e_A = e_A \Rightarrow C_h(e_A) = 0 . \qquad \square$$

Thus we get $C := C_1 \circ \dots \circ C_n$ satisfies

If n is even $C(e_A) = \begin{cases} e_A & \text{if } A = \phi \\ 0 & \text{o/w} \end{cases}$

If n is odd $C(e_A) = \begin{cases} e_A & \text{if } A = \phi \text{ or } \{1, \dots, n\} \\ 0 & \text{o/w} \end{cases}$

If $x \in$ Center of Cl_n then $C_h(x) = x \; \forall h$, and hence $C(x) = x$. The calculation aboves proves that

$$\text{Center of } Cl_n = \begin{cases} \mathbb{R} e_0 & \text{if } n \text{ is even} \\ \mathbb{R} e_0 + \mathbb{R} f & \text{if } n \text{ is odd} \end{cases}$$

Note that for $A, B \subset \{1, \dots, n\}$, $e_A e_B = (\text{sign}).\, e_{\underbrace{A \Delta B}_{\hookleftarrow (A \cup B) \setminus (A \cap B)}}$

Ideals of Cl_n: <u>n even:</u> Let $\mathfrak{a} \subset Cl_n$ be an ideal, $\mathfrak{a} \neq \{0\}$. Then

$\exists \underset{\underset{0}{\#}}{\sum c_A e_A} \in \mathfrak{a}$, say $c_{A_0} \neq 0$. Multiply by e_{A_0} to get coefficient of $e_0 \neq 0$

Apply $C = C_1 \circ \dots \circ C_n$ to get $e_0 \in \mathfrak{a}$ and hence $e_A \in \mathfrak{a}\ \forall A \subset \{1, \dots, n\}$

$\Rightarrow \mathfrak{a} = Cl_n$.

<u>n odd</u>: The argument is similar. If $x = \sum c_A e_A \in \mathfrak{a}$, $c_{A_0} \neq \phi$ for some $A_0 \subset \{1, \dots, n\}$, then $C(e_{A_0} \cdot x) \in \mathfrak{a} \cap \underset{\uparrow \text{ center}}{Z(Cl_n)} = \mathbb{R} e_0 + \mathbb{R} f$.

Note: $\forall A \subset \{1, \dots, n\}$ $\quad e_A^2 = (-1)^{\frac{k(k+1)}{2}} e_0 \quad (k = |A|)$.

Thus $f^2 = (-1)^{n(n+1)/2}$. Let j be a scalar s.t. $j^2 = (-1)^{n(n+1)/2}$. Then

$$u = \frac{e_0 + jf}{2} \qquad v = \frac{e_0 - jf}{2}$$

are orthogonal idempotents of $Z(Cl_n)$

(i.e. $uv = 0$ $\quad u^2 = u$, $v^2 = v$)
$u + v = e_0$

$\Rightarrow$ in case of n odd, the options for $\mathfrak{a}$ are $\{0\}$, $Cl_n \cdot u$, $Cl_n \cdot v$, Cl_n.

□

(17.2) ~~(15.2)~~ $\forall x \in Cl_n$, define $\underset{\swarrow \text{ left mult. by } x}{\mathcal{L}_x} : Cl_n \to Cl_n$
$\qquad y \mapsto x \cdot y$

$2^n \times 2^n$ - matrix w/ $\mathbb{R}$ entries

$$\Delta(x) := \det(\mathcal{L}_x) \in \mathbb{R}.$$

If n is odd, $Cl_n = \underset{\underset{\text{ideal } Cl_n \cdot u}{\parallel}}{Cl_n'} + \underset{\underset{\text{ideal } Cl_n \cdot v}{\parallel}}{Cl_n''} \Rightarrow \mathcal{L}_x = \left[\begin{array}{c|c} \mathcal{L}_x' & 0 \\ \hline 0 & \mathcal{L}_x'' \end{array}\right]$

$\Delta'(x) := \det(\mathcal{L}'_x) \qquad \Delta''(x) := \det(\mathcal{L}''_x)$

Note: for $n \equiv 1\ (4)$ $\quad Cl_n = Cl_n' \oplus Cl_n''$ splits over $\mathbb{C}$, so $\Delta'(x)$, $\Delta''(x)$ are complex-valued

Easy Lemma. $x \in Cl_n$ is invertible $\iff \Delta(x) \neq 0$.

Let $Cl_n^{\times}$ = group of invertible elements of Cl_n. $Cl_n^{\times}$ acts on Cl_n by conjugation, denoted by Ad:

$$x \in Cl_n^{\times} \rightsquigarrow Ad(x) : \begin{array}{l} Cl_n \longrightarrow Cl_n \\ y \longmapsto x y x^{-1} \end{array}$$

(17.3)
~~(15.3)~~ Definition Let M_1 = $\mathbb{R}$-span of $\{e_1, \dots, e_n\}$. Let G be the group of $x \in Cl_n^{\times}$ s.t.
$$\begin{cases} Ad(x)(M_1) \subset M_1 \\ \Delta(x) = 1 \quad (\text{for } n \text{ odd} : \Delta'(x) = \Delta''(x) = 1) \end{cases}$$

Spin$(n) \subset G$ is defined as the component of e_0 (identity). in G.

Remark: The manifold structure on Cl_n (and its subspaces) comes from the embedding $x \longmapsto \mathcal{L}_x$ of Cl_n into $M_{2^n \times 2^n}(\mathbb{R})$.

(17.4)
~~(15.4)~~ Theorem: $\text{Spin}(n) \xrightarrow{\eta} SO(n)$ is a finite covering group. Ker $\eta \neq \{1\}$. is finite.

(i.e. $SO(n) \cong \text{Spin}(n) / \text{finite normal subgp.}$)

non-trivial

Cor. Since Spin(n) is connected by definition, we get that

Ker(η) is a quotient of $\pi_1(SO(n)) \leftarrow$ at most of order 2. [Lemma (14.6) of Lecture 14 page 4]

As Ker$(\eta) \neq \{1\}$, it must be exactly of order 2, hence η is universal covering $\Rightarrow \pi_1(SO(n)) = \mathbb{Z}/_{2\mathbb{Z}} \quad \forall\, n \geq 3$

Spin(n) is compact (finite cover of cpct space)

and simply-connected

(17.5)
~~(15.5)~~ Proof of Theorem. Definition of η: $\forall\, \sigma \in$ Spin(n), let $\eta(\sigma)$

$= Ad(\sigma)\big|_{M_1} \in M_{n\times n}(\mathbb{R})$. Note that $Ad(\sigma)(Y_1 \cdot Y_2) = Ad(\sigma)(Y_1)\cdot Ad(\sigma)(Y_2)$ $\forall\, Y_1, Y_2 \in Cl_n$.

for $x \in M_1$, $x = \sum_{i=1}^{n} x_i e_i$ we have $x^2 = -e_0 \sum_{i=1}^{n} x_i^2$. More

generally, $x = \sum x_i e_i \quad y = \sum y_i e_i \in M_1$

$$\Rightarrow \quad xy = -e_0\left(\sum_{i=1}^{n} x_i y_i\right) + \text{other terms}$$

$\uparrow$ must be preserved by $Ad(\sigma)$

$\Rightarrow Ad(\sigma)\big|_{M_1}$ preserves the standard inner product on $\mathbb{R}^n$

$\Rightarrow \eta(\sigma) \in O(n)$. But Spin(n) is connected and $Id = \eta(e_0)$

$\Rightarrow \eta :$ Spin(n) $\longrightarrow SO(n)$ (component of Id in $O(n)$).

(17.6)
~~(15.6)~~ Surjectivity of η: $M_2 :=$ $\mathbb{R}$-span of $\{e_i e_j : 1 \leq i < j \leq n\}$

is $\frac{n(n-1)}{2}$-dim'l space

Idea is to view M_2 = Lie algebra of Spin(n) and identify it with $\mathfrak{so}(n)$ = Lie alg. of SO(n). ($n \times n$ skew-symmetric matrices).

(1) Since $\eta(\sigma)$ = Conjugation, its differential is ad. Namely, M_2 acts on M_1 by $x \in M_2$, $ad(x)\cdot y = xy - yx$ $\forall y \in M_1$.

Easy check $$[e_ie_j, e_k] = \begin{cases} 0 & \text{if } k \neq i,j \\ 2e_j & \text{if } k = i \\ -2e_i & \text{if } k = j \end{cases}$$ skew-symm. matrix.

Thus differential of η identifies $M_2 \simeq \mathfrak{so}(n)$.

(2) $\forall x \in Cl_n$ we can define $\exp(x) = \sum_{m \geq 0} \frac{x^m}{m!}$ as usual.

Claim: $\exp : M_2 \longrightarrow \mathrm{Spin}(n)$

Proof: $\forall e_ie_j \in M_2$, $$\mathrm{Tr}(\mathcal{L}_{e_ie_j}) = \mathrm{Tr}(\mathcal{L}_{e_i}\mathcal{L}_{e_j}) = \mathrm{Tr}(\mathcal{L}_{e_j}\mathcal{L}_{e_i}) = \mathrm{Tr}(\mathcal{L}_{e_je_i})$$

But $e_ie_j = -e_je_i \Rightarrow \mathrm{Tr}(\mathcal{L}_{e_ie_j}) = 0$ $\forall i<j$. Hence

$\Delta(\exp(e_ie_j)) = e^{\mathrm{Tr}(\mathcal{L}_{e_ie_j})} = 1$. Same argument for Δ', Δ'' if n is odd.

More generally $\forall x \in M_2$, $t \in \mathbb{R}$

$\exp(tx) \in \mathrm{Spin}(n)$ by this calculation, part (1)

and the fact that $\mathrm{Ad}(\exp(tx)) = \exp(t\,ad(x))$

Thus we get

(from - exp is local diffeo Prop.(6.3) of Lecture 6 page 5)

$$\begin{tikzcd} M_2 \arrow[r, equal] \arrow[d, "\exp"'] & so(n) \arrow[d, "\exp"] \\ Spin(n) \arrow[r, "\eta"] & SO(n) \end{tikzcd}$$

⇒ Image of η contains a neighbourhood of Id (covered by exp)

Hence $\mathrm{Im}(\eta) = SO(n)$ (only subgroup containing a neighbourhood of identity in a connected group is the group itself).

(17.7)
~~(15.7)~~

<u>Ker(η) is finite and non-trivial</u> : Note : $(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = - e_1^2 e_2^2 = -e_0$

$$\Rightarrow \quad \exp(t\, e_1 e_2) = \cos(t)\, e_0 + \sin(t)\, e_1 e_2 \in Spin(n) \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \quad -e_0 = \exp(\pi\, e_1 e_2) \in Spin(n)$$

and $-e_0 \in Ker(\eta) \Rightarrow Ker(\eta)$ is non-trivial ($\{\pm e_0\} \subset Ker(\eta)$).

To prove finiteness, we prove that $G \cap Z(Cl_n)$ is finite

n even : if $a \in G \cap Z(Cl_n)$ then $a = a_0 e_0$ ($a_0 \in \mathbb{R}$) and then

$$\Delta(a) = a_0^{2^n} = 1 \quad \Rightarrow \quad a_0 = \pm 1 .$$

n odd : let $x = a e_0 + b f \in G \cap Z(Cl_n)$.

$n \equiv 1$ (4) $\quad u = \dfrac{e_0 + if}{2} \quad v = \dfrac{e_0 - if}{2} \quad \Rightarrow a e_0 + b f = (a - bi)u + (a + bi)v$

($i = \sqrt{-1}$)

Then $\Delta'(x) = (a - bi)^{2^{n-1}} = 1 \quad \Delta''(x) = (a + bi)^{2^{n-1}} = 1 \quad \Rightarrow \quad a + bi$ is $(2^n)^{th}$ root of 1 (finitely many!)

$n \equiv 3$ (4) $\quad u = \dfrac{e_0 + f}{2} \quad v = \dfrac{e_0 - f}{2} \quad \Rightarrow a e_0 + b f = (a+b)u + (a-b)v$

Then $\Delta'(x) = (a+b)^{2^{n-1}} = 1 \quad \Delta''(x) = (a-b)^{2^{n-1}} = 1 \quad \Rightarrow \quad a = \pm 1,\ b = 0$ or $a = 0,\ b = \pm 1$

(again finitely many)

Now $Ker(\eta) \subset G \cap Z(Cl_n)$ hence finite

$\square$