Lecture 18

Classification of root systems

(18.0) Recall : E is a finite-dim'l real vector space together with a positive definite (symmetric bilinear) form $(\cdot,\cdot) : E \times E \to \mathbb{R}$

$R \subset E^* \setminus \{0\}$ finite set . $\begin{array}{ccc} E^* & \xrightarrow{\sim} & E \\ \cup\!\!\!\!\!\!\!\!\;\; & & \cup \\ \gamma & \longmapsto & t_\gamma \end{array}$: $\forall h \in E$, $(t_\gamma, h) = \gamma(h)$

Set $h_\alpha = \frac{2}{(\alpha,\alpha)} t_\alpha \in E$.

(1) R spans E^*. $\alpha, c\alpha \in R \Rightarrow c = \pm 1$

(2) $\forall \alpha, \beta \in R$, $\beta(h_\alpha) \in \mathbb{Z}$. Define $s_\alpha \in GL(E^*)$ (or $GL(E)$) by

$$s_\alpha(\xi) = \xi - \xi(h_\alpha)\alpha \qquad \forall \xi \in E^*$$

$$s_\alpha(x) = x - \alpha(x) h_\alpha \qquad \forall x \in E$$

(3) s_α preserves R $(\forall \alpha \in R)$

(18.1) Let $E^\circ = E \setminus \bigcup_{\alpha \in R} \mathrm{Ker}(\alpha)$. We say $\alpha \in R$ is a wall of C if

- $\overline{C} \cap \mathrm{Ker}(\alpha)$ is of codim 1
- $\alpha(C) > 0$

$\cup$
Choose C a connected component.

Note that for $\alpha \in R$, either $\alpha(x) > 0$ $\forall x \in C$ or $\alpha(x) < 0$ $\forall x \in C$.

$$R_+ = \{\alpha \in R : \alpha(x) > 0 \ \forall x \in C\}$$

$$R_- = \{\alpha \in R : \alpha(x) < 0 \ \forall x \in C\}$$

Clearly $R = R_+ \sqcup R_-$. Since $\forall \alpha \in R$, $-\alpha = s_\alpha(\alpha) \in R$ we get $R_- = -R_+$.

Let $\Delta \subset R_+$ be the set of walls of C. Let us enumerate

$$\Delta = \{\alpha_1, \dots, \alpha_\ell\} \leftarrow \text{Simple roots.}$$

(18.2) Lemma. Let $i \neq j \in \{1, \dots, \ell\}$ and $c > 0$ a real number. Then $\alpha_i - c\alpha_j$ is not a root.

Proof. We know α_i and α_j are not proportional. (Axiom (1) of (18.0))

If $\alpha = \alpha_i - c\alpha_j \in R$, then α is either ≥ 0 or ≤ 0 on $\bar{C}$.

But α on $\operatorname{Ker}\alpha_i \cap \bar{C}$ is < 0 and on $\operatorname{Ker}\alpha_j \cap \bar{C}$ is > 0.

Contradiction! □

Cor. 1 $\forall\, i \neq j \in \{1, \dots, \ell\}$, $a_{ij} := \alpha_j(h_{\alpha_i}) \leq 0$.

Pf. Otherwise (if $a_{ij} > 0$), $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \in R$ will contradict the lemma. □

Cor. 2. Δ is linearly independent

Pf. If $0 = \sum_{t=1}^{p} c_t \alpha_{i_t}$ is a linear relation with all positive coeffs and one of c_t's $\neq 0$, say $c_0 = 1$, we get $-\alpha_{i_0} = \sum_{t=1}^{p} c_t \alpha_{i_t}$

LHS < 0 on C and RHS ≥ 0 on C - contradiction.

Hence any relation if existed must be of the form

$$\beta = \sum_{j=1}^{p} c_j \alpha_{i_j} = \sum_{s=p+1}^{q} d_s \alpha_{i_s}$$

(c's, d's > 0) $\alpha_{i_1} \dots \alpha_{i_p}$ distinct from $\alpha_{i_{p+1}}, \dots, \alpha_{i_q}$

But then $(\beta, \beta) = \sum_{j,s} c_j d_s (\alpha_{i_j}, \alpha_{i_s}) \underset{\text{by Cor 1}}{\leq 0}$

contradicts positive definiteness □

Cor 3. Δ spans E^*

Pf. If $\alpha \in R$ is independent of Δ, $\{\alpha, \alpha_1, \dots, \alpha_\ell\}$ will be a l.i. set

$\Rightarrow \exists\, x \in E$, $y \in E$ s.t. $\alpha_i(x), \alpha_i(y) > 0$, $\alpha(x) > 0$, $\alpha(y) < 0$ Contradiction! □

(18.3) Let $W = \langle s_\alpha : \alpha \in R \rangle$ (Weyl group). Since $W \subset$ Permutations (R) ③

W is necessarily finite. Moreover W preserves $(\cdot,\cdot)$ on E ($\&$ E^*).

(Pf. $(s_\alpha(\xi), s_\alpha(\eta)) = (\xi,\eta) - \xi(h_\alpha)(\alpha,\eta) - \eta(h_\alpha)(\xi,\alpha)$
$$+ \xi(h_\alpha)\,\eta(h_\alpha)\,(\alpha,\alpha)$$

But $(\alpha,\eta) = \eta(h_\alpha)\dfrac{(\alpha,\alpha)}{2}$ and similarly (ξ,α)

$$\Rightarrow (s_\alpha(\xi), s_\alpha(\eta)) = (\xi,\eta) \qquad \square \quad)$$

(18.4) Rank 2 classification.

If $\dim E = 2$, let $\Delta = \{\alpha_1,\alpha_2\}$ and $a_{ij} = \alpha_j(h_i)$ $(i,j = 1 \text{ or } 2)$

$A = \begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}$, $a,b \in \mathbb{Z}_{\geq 0}$. Then $s_1(\alpha_1) = -\alpha_1$, $s_1(\alpha_2) = \alpha_2 + a\alpha_1$, $s_2(\alpha_1) = \alpha_1 + b\alpha_2$, $s_2(\alpha_2) = -\alpha_2$

In matrix form $s_1 = \begin{bmatrix} -1 & a \\ 0 & 1 \end{bmatrix}$ $s_2 = \begin{bmatrix} 1 & 0 \\ b & -1 \end{bmatrix}$ $\Rightarrow s_1 s_2 = \begin{bmatrix} -1+ab & -a \\ b & -1 \end{bmatrix}$

Characteristic poly. of $s_1 s_2$: $(T+1-ab)(T+1) + ab$
$$= T^2 + (2-ab)T + 1$$

$s_1 s_2$ has finite order $\Rightarrow$ roots of characteristic poly have modulus 1

so $ab = 0, 1, 2$ or 3 (Note: $ab = 4$ will contradict l.i. of α_1, α_2)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$A_1 \times A_1$ $\qquad A_2$ $\qquad B_2$ (or C_2) $\qquad G_2$

(18.5) Let $A = \left(a_{ij} = \alpha_j(h_i)\right)_{1 \le i,j \le \ell}$ [Cartan Matrix]

Dynkin diagram Γ_A : vertices $\{1, \dots, \ell\}$. Edges, b/w i & j

$i \circ \quad \bullet j$: $a_{ij} = 0 = a_{ji}$

$i \bullet\!\!-\!\!-\!\!\bullet j$: $a_{ij} = a_{ji} = -1$

$i \bullet \Leftarrow \bullet j$: $a_{ij} = -2$, $a_{ji} = -1$

$i \bullet \Lleftarrow \bullet j$: $a_{ij} = -3$, $a_{ji} = -1$

Assume Γ_A is connected (i.e. A is not of the form $\left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$ after rearranging $\{1, \dots, \ell\}$)

The classification of Γ_A rests on the following fact – a consequence of positive definiteness of $(\cdot,\cdot)$ whose matrix is $D \cdot A$ when D is diagonal, $d_i = \frac{(\alpha_i, \alpha_i)}{2} > 0$.

If $\underline{u} \in \mathbb{R}^\ell$ is s.t. $\begin{matrix} \underline{u} \ge 0 \\ A\underline{u} \le 0 \end{matrix}$ then $\underline{u} = 0$ ($\underline{u} \ge 0$ means all coordinates of $\underline{u}$ are ≥ 0) $\longrightarrow$ $(*)$

Now we just exclude cases which contradict $(*)$. For example

- if $\bullet \Lleftarrow \bullet \cdots\cdots \bullet$ (edge of any type) is a subdiagram of Γ_A then $\begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & -a \\ 0 & -b & 2 \end{bmatrix}$ is part of A

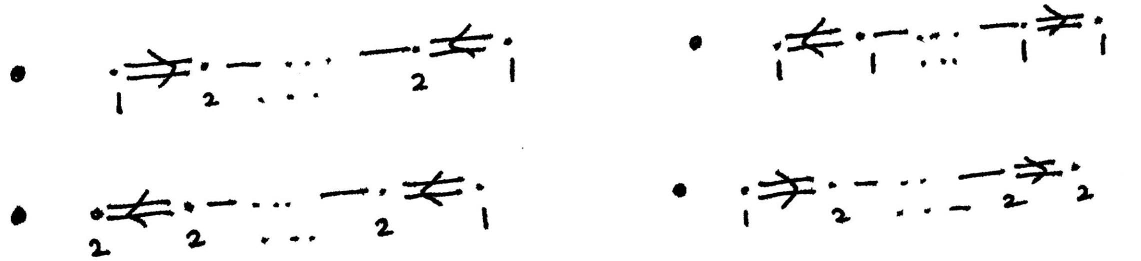
and $\underline{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ on these 3 vertices and 0's everywhere else

contradicts $(*)$: $\begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & -a \\ 0 & -b & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1-a \\ 2-2b \end{bmatrix} \le 0$.

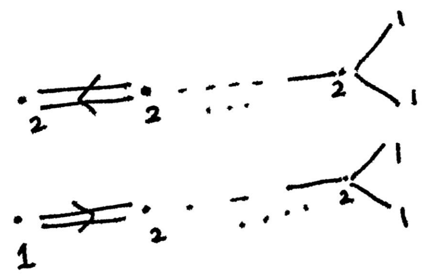
Similarly $\bullet \cdots \bullet \Lleftarrow \bullet$ is excluded using $\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow$ If Γ_A contains $\bullet \Lleftarrow \bullet$ then $\Gamma_A = \bullet \Lleftarrow \bullet$ is of type G_2 (by connectedness)

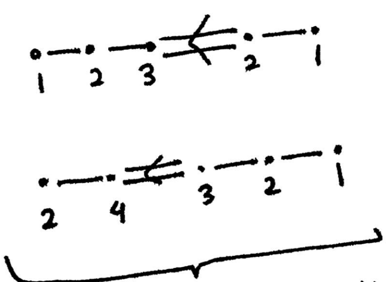
From now on I will just write the coordinates of $\underline{u}$ on vertices which contradicts $(*)$



Γ_A cannot have 2 edges of type $\Leftarrow$.



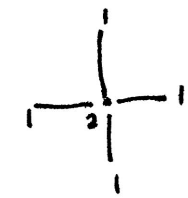
If Γ_A contains $\Leftarrow$ then it cannot have a vertex of degree ≥ 3



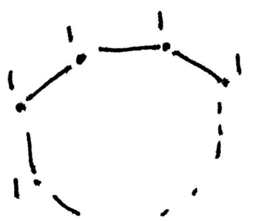
If Γ_A contains $\Leftarrow$ then it is of type F_4, B_n or C_n

→ Now assume all edges of Γ_A are •——•

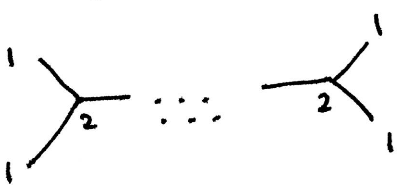
Again



$\Rightarrow$ Γ_A cannot have a vertex of degree ≥ 4

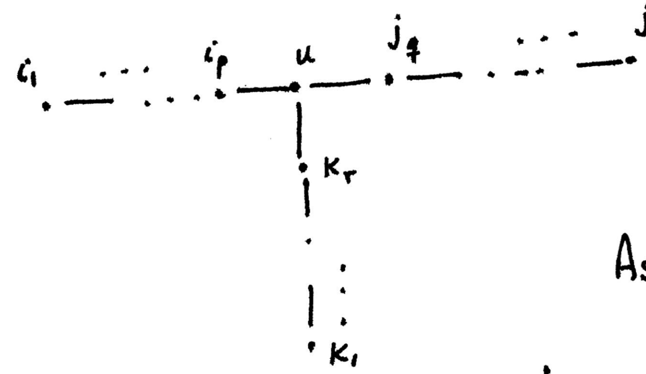


$\Rightarrow$ Γ_A cannot have a cycle.



$\Rightarrow$ Γ_A cannot have 2 vertices of degree 3.

Thus the only possibility is



$(p, q, r \geq 0)$

Assume $r \leq q \leq p$

In this case A is symmetric, can be taken as the matrix of $(\cdot,\cdot)$

Set $x = \sum_{t=1}^{p} t\,\alpha_{i_t}$ $\quad y = \sum_{t=1}^{q} t\,\alpha_{j_t}$ $\quad z = \sum_{t=1}^{r} t\,\alpha_{k_t}$ $\quad w = \alpha_u$

Then x, y, z are orthogonal to each other

$$|x|^2 = p(p+1) \qquad |y|^2 = q(q+1) \qquad |z|^2 = r(r+1)$$

$$(x,w) = -p \qquad (y,w) = -q \qquad (z,w) = -r \qquad (w,w) = |w|^2 = 2.$$

$(\text{Distance})^2$ between w and the $\mathbb{R}$-span of $\{x,y,z\}$ is given by

$$|w|^2 - \frac{(x,w)^2}{|x|^2} - \frac{(y,w)^2}{|y|^2} - \frac{(z,w)^2}{|z|^2} > 0 \qquad (\text{as } w \notin \mathbb{R}\text{-span of } x,y,z)$$

$$\Rightarrow \quad 2 - \frac{p}{p+1} - \frac{q}{q+1} - \frac{r}{r+1} > 0 \quad \Rightarrow \quad \frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} > 1$$

- L.H.S. is $< \dfrac{3}{r+1}$ $\Rightarrow$ $\dfrac{3}{r+1} > 1$ $\Rightarrow$ $r < 2$

  (as $r \le q \le p$ assumed)

  so $\underbrace{r = 0}_{\downarrow}$ or 1

  no constraint on p, q $\rightsquigarrow$ type A.

If $r = 1$ we get $\dfrac{1}{p+1} + \dfrac{1}{q+1} > \dfrac{1}{2}$ $\qquad$ $q = 1 \Rightarrow$ no constraint on p $\rightsquigarrow$ Type D

$(p \ge q \ge 1)$

Again $\dfrac{2}{q+1} \geqslant \dfrac{1}{p+1} + \dfrac{1}{q+1} > \dfrac{1}{2}$ $\Rightarrow$ $q < 3$, so $q = 1$ or 2

If $q = 2$ we get $\dfrac{1}{p+1} > \dfrac{1}{6}$ $\Rightarrow$ $2 \le p < 5$

$(p \ge 2)$

so $p = \underbrace{2}_{E_6}, \underbrace{3}_{E_7}$ or $\underbrace{4}_{E_8}$.

$\square$