Lecture 19

(19.0) Recall: $R \subset E^* \setminus \{0\}$ is a root system. Let $\mathcal{C}$ = set of connected components of $E \setminus \bigcup_{\alpha \in R} \mathrm{Ker}(\alpha)$. We fixed a $C_0 \in \mathcal{C}$ (fundamental chamber) and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset R$ (simple roots) is the set of walls of C_0

$$\underset{\text{(Weyl group)}}{W} = \langle s_\alpha : \alpha \in R \rangle \underset{\substack{\text{finite}\\ \text{subgp.}}}{\subset} GL(E) \text{ or } GL(E^*)$$

(19.1) **Lemma.** Let $x \in E$ and $W' \subset W$ be subgroup generated by $\{s_i = s_{\alpha_i}\}_{i=1,\ldots,\ell}$. Then $W' \cdot x \cap \overline{C_0} \neq \phi$.

Proof. Choose a point $a \in C_0$ and let $y \in W' \cdot x$ be such that $d(y, a) \leq d(z, a) \quad \forall z \in W'x$.

(Here $d(a,b)^2 = \|a-b\|^2 = (a-b, a-b)$ distance defined by positive-definite form). For each $i \in \{1, \ldots, \ell\}$ we have

$$d(y,a)^2 \leq d(s_i y, a)^2 \iff (y-a, y-a) \leq (y - \alpha_i(y) h_i - a, y - \alpha_i(y) h_i - a)$$

$$\iff 0 \leq \alpha_i(y)^2 (h_i, h_i) - 2\alpha_i(y)(y-a, h_i)$$

$$\iff 0 \leq \alpha_i(y)^2 \frac{4}{(\alpha_i, \alpha_i)} - \frac{4\alpha_i(y)(\alpha_i(y) - \alpha_i(a))}{(\alpha_i, \alpha_i)} \qquad \text{since } (x, h_i) = \frac{2\alpha_i(x)}{(\alpha_i, \alpha_i)} \;\; \forall x \in E$$

$\iff \; 0 \leq \alpha_i(y)\,\alpha_i(a)$. But $\alpha_i(a) > 0$ since $a \in C_0$

so we get $\alpha_i(y) \geq 0 \;\; \forall \; i = 1, \dots, \ell$. Hence $y \in \overline{C}_0$ $\square$

(19.2) Corollaries of Lemma ((19.1))

(i) $W' = W$ (i.e. W is generated by $\{s_i\}_{i=1,\dots,\ell}$)

For any $\alpha \in R$, let $C \in \mathcal{C}$ be a chamber adjacent to $\mathrm{Ker}(\alpha)$. Since $W' \cdot C \cap \overline{C}_0 \neq \phi$ we must have $wC = C_0$ for some $w \in W'$. Hence $w(\alpha) = \alpha_i$ for some $i \in I$ and hence $S_\alpha = S_{w^{-1}\alpha_i}$ $= w^{-1} s_i w \in W'$ proving that $W = W'$ $\square$. Also we proved that

(ii) $R = \bigcup_{i=1,\dots,\ell} W\alpha_i$

(iii) $R_+ = R \cap \left(\sum_{i=1}^{\ell} \mathbb{N}\alpha_i \right)$ $\quad R_- = -R_+$

We already know that: $R_+ \ni \alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ with $c_i \geq 0 \;\; \forall i$ (real)

From (ii) α is also obtained from some α_j by applying simple reflections $\{s_1, \dots, s_\ell\}$. Since at any stage $s_k(\alpha_{k'}) \in \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i$

we get that $c_i \in \mathbb{Z} \;\; \forall i$ $\square$

(19.3) Recall that $R_+ = \{\alpha \in R \,:\, \alpha(C_0) > 0\}$. We just proved that $W = \langle s_i : i = 1, \dots, \ell \rangle$. Now $\forall \; i,j \in \{1, \dots, \ell\}$

let m_{ij} = order of $(s_i \cdot s_j)$. From rank 2 classification we have

$a_{ij} \cdot a_{ji}$	0	1	2	3
m_{ij}	2	3	4	6

Length function: for $w \in W$, let $\ell(w)$ be the smallest k s.t. $w = s_{i_1} \cdots s_{i_k}$. If this is the case $w = s_{i_1} \cdots s_{i_k}$ is called a reduced expression for w.

(19.4) Prop. (i) Let $\alpha \in R_+$ and $i \in \{1, \dots, \ell\}$. Then $s_i \alpha \in R_- \iff \alpha = \alpha_i$

(ii) The following are equivalent for $w \in W$ and $i \in \{1, \dots, \ell\}$

(a) $\ell(w s_i) < \ell(w)$

(b) $w(\alpha_i) \in R_-$

(c) [Exchange Property] for any reduced exp. $w = s_{i_1} \cdots s_{i_k}$,

$\exists\, j,\ 1 \leq j \leq k$ s.t. $s_{i_j} \cdots s_{i_k} = s_{i_{j+1}} \cdots s_{i_k} \cdot s_i$

Proof of (i): $\left.\begin{array}{l} s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i \in R_- \\ \alpha \in R_+ \end{array}\right\} \Rightarrow \alpha$ and α_i are proportional

and hence (as $\alpha \in R_+$), $\alpha = \alpha_i$ $\square$

Proof of (ii): (c) $\Rightarrow$ (a) clear

(b) $\Rightarrow$ (c) Let $\beta_j := s_{i_{j+1}} \cdots s_{i_k} \alpha_i$ $(0 \leq j \leq k)$.

Since $\beta_0 \in R_-$ and $\beta_k \in R_+$, there must exist j s.t.

$\beta_j \in R_+$, $\beta_{j-1} \in R_-$. But $\beta_{j-1} = s_{i_j}(\beta_j)$ and by (i)

we get $\beta_j = \alpha_{i_j}$, i.e. $\alpha_{i_j} = s_{i_{j+1}} \cdots s_{i_k}(\alpha_i)$ and hence

$$s_{i_j} = s_{i_{j+1}} \cdots s_{i_k} s_i s_{i_k} \cdots s_{i_{j+1}} \Rightarrow \text{Exchange Prop.} \quad \square$$

(a) $\Rightarrow$ (b). Assume $\ell(ws_i) < \ell(w)$ and $w(\alpha_i) \in R_+$.

Then $ws_i(\alpha_i) \in R_-$ and by (b) $\Rightarrow$ (c) $\Rightarrow$ (a) applied to ws_i

we get $\ell(w\underbrace{s_i s_i}_{s_i^2=1}) < \ell(ws_i)$, a contradiction $\square$