Lecture 20

(20.0) Our next theorem is Peter-Weyl Theorem. For a compact Lie group G, the theorem states

$$\underset{\substack{\uparrow \\ \text{square-integrable} \\ \mathbb{C}\text{-valued fns. on } G}}{L^2(G)} \simeq \bigoplus_{\lambda \in \Lambda} V_\lambda^{\oplus m_\lambda}$$

$\{V_\lambda\}_{\lambda \in \Lambda}$ set of iso. classes of irreducible finite-dim'l representations of G

m_λ = multiplicity of V_λ = $\dim V_\lambda$

In order to define L^p-spaces associated to G, we need a measure on G (wrt Borel σ-algebra), or what is equivalent, by Riesz representation theorem, an integral

$$\int_G : \underset{\substack{\text{(continuous} \\ \mathbb{C}\text{-valued fns.} \\ \text{on } G)}}{C(G)} \longrightarrow \mathbb{C}$$

We want this integration to be invariant under left-translations of G. This will lead us to constructing invariant differential forms on G.

(20.1) Recall : for an m-dim'l smooth manifold M, we defined TM (tangent bundle) and T^*M (cotangent bundle)

A 1-form ω on M is a section of $\begin{array}{c} T^*M \\ \downarrow \pi \\ M \end{array}$, i.e. a smooth function $\omega : M \longrightarrow T^*M$ s.t. $\pi \circ \omega = \mathrm{Id}_M$.

For any r, $1 \le r \le m$, a (smooth) r-form on M is a section of $\begin{array}{c} \Lambda^r T^*M \\ \downarrow \\ M \end{array}$ (r^{th} exterior product). We usually denote by

$\Omega^r(M)$ = space of r-forms on M.

(module over $C^\infty(M)$ - algebra of smooth functions $M \to \mathbb{R}$)

Locally, for $p \in M$ and a coordinate neighbourhood $\mathcal{U} \xrightarrow[(x_1, \dots x_m)]{\sim} \begin{array}{c} \mathrm{Cube}_r(\underline{0}) \\ \cap \\ \mathbb{R}^m \end{array}$

$dx_1, \dots, dx_m$ are 1-forms on $\mathcal{U}$

For $1 \le i_1 < \dots < i_r \le m$, $dx_{i_1} \wedge \dots \wedge dx_{i_r}$ is an r-form on $\mathcal{U}$.

Change of coordinates, say near p we have 2 coordinates

$$p \in \mathcal{U} \begin{array}{l} \xrightarrow[(x_1, \dots x_m)]{\sim} \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^m \\ \xrightarrow[(y_1, \dots y_m)]{\sim} \mathrm{Cube}_{r'}(\underline{0}) \subset \mathbb{R}^m \end{array}$$

We get $y_j = F_j(x_1, \ldots, x_m)$ $(\forall\ 1 \le j \le m)$. Here $F_1, \ldots, F_m$ are $\mathbb{R}$-valued fns. of m real variables and $\forall q \in \mathcal{U}$

$$y_j(q) = F_j(x_1(q), \ldots, x_m(q)).$$

Thus $dy_j = \sum_{i=1}^{m} \frac{\partial F_j}{\partial x_i} dx_i$. Similarly for an r-form

$$dy_{j_1} \wedge \ldots \wedge dy_{j_r} = \left(\sum_{i=1}^{m} \frac{\partial F_{j_1}}{\partial x_i} dx_i \right) \wedge \ldots \wedge \left(\sum_{i=1}^{m} \frac{\partial F_{j_r}}{\partial x_i} dx_i \right)$$

$$= \sum_{1 \le i_1 < \ldots < i_r \le m} \left\{ \sum_{\sigma \in S_r} (-1)^{\sigma} \frac{\partial F_{j_{\sigma(1)}}}{\partial x_{i_1}} \cdots \frac{\partial F_{j_{\sigma(r)}}}{\partial x_{i_r}} \right\} dx_{i_1} \wedge \ldots \wedge dx_{i_r}$$

(use $dx_i \wedge dx_j = -dx_j \wedge dx_i$)

In particular, for $r = m$, we get

$$dy_1 \wedge \ldots \wedge dy_m = \det \left(\frac{\partial F_i}{\partial x_j} \right)_{1 \le i, j \le m} dx_1 \wedge \ldots \wedge dx_m$$

(20.2) Orientability. An orientation on M is a choice of ordering the coordinate functions on coordinate neighbourhoods of M s.t. the choices on overlapping coordinate open sets are consistent. That is, for every $\mathcal{U} \subset M$, with homeo. $\mathcal{U} \xrightarrow{\sim} \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^m$ we fix an order on the m-components of this homeo. (equivalently an ordered basis of $\mathbb{R}^m$).

If $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \phi$

$$\begin{array}{ccc} & & \mathcal{U}_1 \xrightarrow{x_1 \dots x_m} \mathrm{Cube}_r(\underline{0}) \\ & \nearrow & \\ \mathcal{U}_1 \cap \mathcal{U}_2 & & \\ & \searrow & \\ & & \mathcal{U}_2 \xrightarrow{y_1 \dots y_m} \mathrm{Cube}_{r'}(\underline{0}) \end{array}$$

We require $\det\left(\dfrac{\partial y_j}{\partial x_i}\Bigg|_q\right) > 0 \quad \forall\, q \in \mathcal{U}_1 \cap \mathcal{U}_2$.

More systematically, an orientation on M is a choice of nowhere vanishing m-form, say $\phi \in \Omega^m(M)$. Such a choice determines an ordering (up to even permutations) of coordinate functions $\mathcal{U} \xrightarrow[x_1 \dots x_m]{} \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^m$ by: $x_1 \dots x_m$ is oriented (wrt. ϕ) if

$$dx_1 \wedge \dots \wedge dx_m = F \cdot \phi\big|_{\mathcal{U}} \quad \text{and} \quad F(q) > 0 \quad \forall\, q \in \mathcal{U}.$$

Examples. Spheres are orientable. $\mathbb{P}^2$ is <u>not</u> orientable.

All Lie groups are orientable as we will see later.

For an orientable manifold M, with a choice of nowhere vanishing m-form ϕ, we can define integration of compactly supported continuous ($\mathbb{R}$-valued, or $\mathbb{C}$-valued) functions on M.

(20.3) M, ϕ : oriented Manifold of dimension m.

$$C_c(M) = \left\{ f : M \to \mathbb{R} \text{ continuous such that } \overline{\mathrm{Supp}(f)} \text{ is compact} \right\}$$

Here $\mathrm{Supp}(f) = \{ p \in M : f(p) \neq 0 \}$

Definition of $\int_M f\phi$: Step 1. Assume $\mathrm{Supp}(f) \subset$ a coord. nhd.

say $\mathcal{U} \xrightarrow[x_1, \dots, x_m]{\sim} \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^m$. Then we get two real-valued

ordered via ϕ

functions of m real variables $\tilde{f}$ and F such that $\forall q \in \mathcal{U}$

$$f(q) = \tilde{f}(x_1(q), \dots, x_m(q))$$

$$\phi(q) = F(x_1(q), \dots, x_m(q)) \; dx_1 \wedge \dots \wedge dx_m \big|_q$$

(here $F > 0$).

$$\int_M f\phi := \int_{\mathrm{Cube}_r(\underline{0})} \tilde{f}(u_1 \dots u_m)\, F(u_1 \dots u_m)\, du_1 \dots du_m$$

Use change of coordinate formulae and change of variables for integrals to prove independence from the choice of $(x_1, \dots, x_m)$:

$\mathcal{U} \xrightarrow{\sim} \mathrm{Cube}_r(\underline{0})$

Step 2. By step 1 we know how to integrate functions $f \in C_c(M)$ s.t. $\overline{\mathrm{Supp}(f)} \subset$ a coord. nhd. $\mathcal{U}$. We have the usual properties of $\int_M - \phi$ restricted to such functions (i.e. fix a coord. nhd. $\mathcal{U}$ and consider $C_c(M)_{\mathcal{U}} = \{ f \in C_c(M) : \overline{\mathrm{Supp} f} \subset \mathcal{U} \}$)

Then
$$\int_M (a_1 f_1 + a_2 f_2)\, \phi = a_1 \int_M f_1 \phi + a_2 \int_M f_2 \phi.$$
$$\forall\ f_1, f_2 \in C_c(M)_{\mathcal{U}}$$

Now, if $f \in C_c(M)$ is arbitrary, we use the following

Lemma. Let $K \subset M$ be a compact subset. Then there exists a continuous function η on M such that

(1) $\eta = \eta_1 + \dots + \eta_\ell$ finite sum of functions $\eta_j \in C_c(M)_{\mathcal{U}_j}$ with each $\mathcal{U}_j \subset M$ a coord. nhd.

(2) $\eta(q) = 1 \quad \forall\ q \in K.$

Then (assuming the lemma for now), take $K = \overline{\mathrm{Supp}(f)}$ and let η be the continuous function given by the lemma.

Define
$$\int_M f \phi = \sum_{i=1}^{\ell} \int_M f \cdot \eta_i\, \phi$$
(note $f = f \cdot \eta$ since $f = 0$ outside K, $\eta = 1$ inside K)

(20.4) Proof of independence from η.

More generally, if $f = \sum_{j=1}^{r} f_j = \sum_{k=1}^{s} g_k$ where each f_j and g_k have compact support in a coord. open set, we will prove that $\sum_{j=1}^{r} \int f_j \phi = \sum_{k=1}^{s} \int g_k \phi$. To see this, let η be the function coming from Lemma (20.3) above, with

$$K = \bigcup_{j=1\ldots r} \operatorname{Supp}(f_j) \cup \bigcup_{k=1\ldots s} \operatorname{Supp}(g_k)$$

. Then we get (say $\eta = \eta_1 + \dots + \eta_t$)

$$f\eta_i = \sum_j f_j \eta_i = \sum_k g_k \eta_i$$

$$\underset{\text{(by Step 1)}}{\Longrightarrow} \quad \sum_j \int f_j \eta_i \phi = \sum_k \int g_k \eta_i \phi$$

Summing over i and using $f_j = \sum_i f_j \eta_i$ (same w/ g's)

we get $\sum_j \int f_j \phi = \sum_k \int g_k \phi$

(20.5) Proof of Lemma (20.3) page 6:

$\underset{\text{compact}}{K \subset M}$. We can cover $K \subset \bigcup_{j=1}^{\ell} V_j =: \mathcal{U}$

$$V_j \xrightarrow[x_1^j \dots x_m^j]{\sim} \mathrm{Cube}_r(\underline{0}) \subset \mathbb{R}^m$$

Define $\mu_j(q) = \begin{cases} 1 - \max\limits_{1 \le i \le m} \dfrac{|x_i^j(q)|}{r} & \text{if } q \in V_j \\ 0 & \text{o/w} \end{cases}$

(by shrinking V_j if necessary we can assume $V_j \subsetneq \tilde{V}_j$ another coord. nhd.)

Let $\mu = \sum_{j=1}^{\ell} \mu_j$ (> 0 on $\mathcal{U}$) continuous fn
($= 0$ on $M \setminus \mathcal{U}$)

Let $m = \min\{\mu(p) : p \in K\} > 0$ since K is cpct.

We just need to average μ. by $s(p) = \mathrm{Max}\{m, \mu(p)\}$
($= \mu(p)$ on K)

Thus $\eta = \dfrac{\mu}{s} = \sum_{j=1}^{\ell} \dfrac{\mu_j}{s} \leftarrow \eta_j$ is the desired function. $\square$