Lecture 21

(21.0) Recall : for a smooth connected m-dim'l manifold M, we defined :

- $\Omega^r(M) = \{ r\text{- forms on } M \}$ $(0 \le r \le m)$
  $=$ smooth sections of $\Lambda^r T^*M$

$(\Omega^0(M) = C^\infty(M))$

- $\forall \, \phi \in \Omega^m(M)$, we have $\int_M - \phi : C_c(M) \longrightarrow \mathbb{R}$

$\mathbb{R}$ - linear functional

$C_c(M)$: continuous $f : M \to \mathbb{R}$ s.t. $\overline{\mathrm{Supp}(f)}$ is compact.

(21.1) de Rham differential (optional)

$$d : \Omega^r(M) \longrightarrow \Omega^{r+1}(M)$$

- Let $M = \bigcup M_\alpha$ be an open covering by coord. open sets

$$M_\alpha \xrightarrow[(x_1^\alpha, \ldots, x_m^\alpha)]{\text{homeo.} \ \sim} \mathrm{Cube}_{r_\alpha}(\underline{0}) \subset \mathbb{R}^m$$

$$\omega \in \Omega^r(M) \longleftrightarrow \omega_\alpha = \omega|_{M_\alpha} \ \forall \alpha \ \text{ s.t.}$$

$$\omega_\alpha|_{M_\alpha \cap M_\beta} = \omega_\beta|_{M_\alpha \cap M_\beta}$$

- Locally $\omega_\alpha = \sum_{1 \le i_1 < \ldots < i_r \le m} f_{\alpha; \underline{i}} \, dx_{i_1}^\alpha \wedge \ldots \wedge dx_{i_r}^\alpha$

where for each $\underline{i} = i_1 < \dots < i_r$, $f_{\alpha;\underline{i}}$ is a smooth function on M_α.

- Consistency on overlaps $M_\alpha \cap M_\beta$: recall that the transition (or change of coordinate) functions are assumed to be smooth functions $F^\beta_{i;\alpha}(x_1 \dots x_m)$ ($\mathbb{R}$-valued of m real variables)

$$x^\beta_i(q) = F^\beta_{i;\alpha}(x^\alpha_1(q), \dots, x^\alpha_m(q)) \quad \forall\, q \in M_\alpha \cap M_\beta$$

$$\Rightarrow \quad dx^\beta_i = \sum_{j=1}^{m} \frac{\partial}{\partial x_j} F^\beta_{i;\alpha} \, . \, dx^\alpha_j$$

We get (Lecture 20, page 3) :

$$dx^\beta_{i_1} \wedge \dots \wedge dx^\beta_{i_r} = \sum_{j_1 < \dots < j_r} \underbrace{\Delta_{\underline{i},\underline{j}}(J^\beta_\alpha)} \, . \, dx^\alpha_{i_1} \wedge \dots \wedge dx^\alpha_{i_r}$$

$$\text{set } \quad J^\beta_\alpha = \left[\frac{\partial}{\partial x_j} F^\beta_{i;\alpha} \right]_{1 \le i,j \le m}$$

$\Delta_{\underline{i},\underline{j}}(J^\beta_\alpha)$ = determinant of $r \times r$ submatrix of J^β_α w/ rows $i_1 \dots i_r$ & cols. $j_1 \dots j_r$

So we get (by an easy computation) $\omega_\alpha = \omega_\beta$ on $M_\alpha \cap M_\beta$

iff $\forall\, (\ell_1 < \dots < \ell_r) =: \underline{\ell}$

$$(*) \quad f_{\alpha;\underline{\ell}}(x^\alpha_1 \dots x^\alpha_m) = \sum_{\substack{\underline{i} = \\ (i_1 < \dots < i_r)}} f_{\beta;\underline{i}}(x^\beta_1 \dots x^\beta_m) \, . \, \Delta_{\underline{i},\underline{\ell}}(J^\beta_\alpha)$$

Thus $\omega \in \Omega^r(M)$ is completely determined by

$$\{ f_{\alpha;\underline{i}} \in C^\infty(M_\alpha) \}_{\alpha,\, i_1 < \dots < i_r}$$ satisfying $(*)$, and conversely.

- Define $d\omega \in \Omega^{r+1}(M)$ locally by

$$d\omega \Big|_{M_\alpha} = \sum_{i_1 < \dots < i_{r+1}} \left\{ \sum_{t=1}^{r+1} (-1)^{t+1} \frac{\partial f_{\alpha;\, i_1 \dots \hat{i}_t \dots i_{r+1}}}{\partial x^\alpha_{i_t}} \right\} dx^\alpha_{i_1} \wedge \dots \wedge dx^\alpha_{i_{r+1}}$$

Easy exercise - consistency equation holds $(*)$. Hence we get $d : \Omega^r(M) \longrightarrow \Omega^{r+1}(M)$.

Another easy exercise: $d^2 = 0$. de Rham complex is:

$$\Omega^*(M) : \quad 0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \to \Omega^m(M) \to 0$$

de Rham cohomology

$$H^i_{dR}(M) := \frac{\operatorname{Ker}\left(\Omega^i(M) \longrightarrow \Omega^{i+1}(M) \right)}{\operatorname{Im}\left(\Omega^{i-1}(M) \to \Omega^i(M) \right)} \qquad (0 \le i \le m)$$

- Compute H^0 and H^1 of S^1 as an ~~ist~~ instructive exercise.

(21.2) Pull back of differential forms.

Let $\eta : M \longrightarrow N$ be a smooth map. Recall that η being smooth means $\forall\, p \in M$, we can find a coordinate neighbourhood of $q = \eta(p) \in N$ and $p \in M$, say $\underset{\substack{\cup\!\!\!\mid \\ p}}{\mathcal{U} \subset M}$, $\underset{\substack{\cup\!\!\!\mid \\ q}}{V \subset N}$ s.t. $\eta^{-1}(V) = \mathcal{U}$ and

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\ \eta\ } & V \\ \wr\downarrow & & \wr\downarrow \\ \mathrm{Cube}_r(\underline{0}) & \dashrightarrow & \mathrm{Cube}_s(\underline{0}) \\ \cap\!\mathrm{l} & & \cap\!\mathrm{l} \\ \mathbb{R}^m & & \mathbb{R}^n \end{array}$$

is smooth fn.

Thus we can cover $M = \bigcup M_\alpha$ ~~and $N = \bigcup N_\alpha$~~ s.t. $\eta(M_\alpha) \subset \underset{\uparrow \text{ coord. nhd.}}{N_\alpha} \subset N$. This allows us to define the pull-back of $\omega \in \Omega^r(N)$, denoted by $\eta^*\omega \in \Omega^r(M)$ locally via:

if $$\omega\big|_{N_\alpha} = \sum_{1 \le i_1 < \dots < i_r \le n} f_{\alpha;\underline{i}}\, dx^\alpha_{i_1} \wedge \dots \wedge dx^\alpha_{i_r}$$

then $$\eta^*\omega\big|_{M_\alpha} := \sum_{\underline{i}} \left(f_{\alpha;\underline{i}} \circ \eta\right) d\left(x^\alpha_{i_1} \circ \eta\right) \wedge \dots \wedge d\left(x^\alpha_{i_r} \circ \eta\right)$$

where $$M_\alpha \xrightarrow{\ \eta\ } N_\alpha \xrightarrow{(x^\alpha_1, \dots, x^\alpha_n)} \mathrm{Cube}_{s_\alpha}(\underline{0}) \subset \mathbb{R}^n$$

One need to check (easy!) that $\{\eta^*\omega|_{M_\alpha}\}$ satisfy $(*)$ of page 2 above.

(21.3) Back to Lie groups. Let G be a connected Lie group of dimension n. An r-form $\omega \in \Omega^r(G)$ is said to be left-invariant if $\forall\, \sigma \in G$ $\quad l_\sigma^* \omega = \omega$. Here $l_\sigma : G \to G$, $x \mapsto \sigma x$ is left mult.

(Prop (1.4) Lecture 1 page 5) $\mathfrak{g} = T_e G$ = left-invariant vector fields on G

Analogously, $\mathfrak{g}^*$ = left-invariant 1-forms on G. If we pick a basis $x_1, \dots, x_n$ of $\mathfrak{g}$, we obtain $dx_1, \dots, dx_n$ (dual basis of $\mathfrak{g}^*$) 1-forms on G.

$\Lambda^n \mathfrak{g}^*$ is 1-dim'l vector space (over $\mathbb{R}$). Thus up to a scalar $\phi = dx_1 \wedge \dots \wedge dx_n$ is the only left-invariant n-form on G.

Note, if $\phi(\sigma) = 0$ for some $\sigma \in G$, then $\phi(\tau) = 0$ $\forall\, \tau \in G$ by invariance. Since we know $\phi \neq 0$ element of $\Lambda^n \mathfrak{g}^*$, we get that ϕ is nowhere vanishing, hence G is orientable.

From now on, we fix ϕ and assume G is oriented via ϕ (see Lecture 20 page 4).

(21.4) Modularity. For $\sigma_0 \in G$, consider $\mathrm{Conj}(\sigma_0) : G \to G$, $x \mapsto \sigma_0 x \sigma_0^{-1}$

Then $\mathrm{Conj}(\sigma_0)^* \phi$ is another left-invariant n-form on G.

$\Rightarrow \exists\ c(\sigma_0) \in \mathbb{R}_{\neq 0}$ s.t. $\mathrm{Conj}(\sigma_0)^* \phi = c(\sigma_0) \cdot \phi$.

In terms of $\int_G$ defined using ϕ, we get the following identities

$$\text{(i)} \quad \int_G (f \circ l_\sigma)\, \phi = \int_G f \cdot \phi \qquad \forall \sigma \in G$$

Let $r_{\sigma_0} : G \to G$ be the right multiplication. Then
$$x \mapsto x\sigma_0$$

$\left(\mathrm{Conj}_{\sigma_0}\right)^{-1} = l_{\sigma_0}^{-1} \circ r_{\sigma_0}$ by definition. And we get

$$\text{(ii)} \quad \int_G (f \circ r_{\sigma_0})\, \phi = \int_G \left(f \circ l_{\sigma_0} \circ \mathrm{Conj}(\sigma_0)^{-1}\right) \phi$$

$$= \int_G (f \circ l_{\sigma_0}) \left(\mathrm{Conj}(\sigma_0)^* \phi\right) = c(\sigma_0) \int_G (f \circ l_{\sigma_0})\, \phi$$

$$= c(\sigma_0) \int_G f \cdot \phi .$$

(21.5) Let G be a compact (connected) Lie group. We normalize ϕ so that $\int_G 1 \cdot \phi = 1$

constant function 1.

From (ii) of page 6, we get $c(\sigma_0) = 1 \quad \forall\, \sigma_0 \in G$.

We fix this n-form (both left & right invariant) and will use the familiar notation $\int_G f(x)\, dx$ for $\int_G f \cdot \phi$.

Note: Lie groups for which $c(\sigma_0) = 1 \quad \forall\, \sigma_0$ are called <u>unimodular</u>. All compact Lie groups are <u>unimodular</u>.

Similar to the computation (ii) of page 6, gives $\int_G f(x^{-1})\, dx = \int_G f(x)\, dx$

(21.6) Consequences of $\int_G f(x)\, dx$:

(a) Let V be a finite-dimensional repn. of G, over $\mathbb{C}$. That is, V is a f.d. vector space over $\mathbb{C}$ and we have a continuous map (group hom.) $G \longrightarrow GL(V)$.

Then V admits a G-invariant Hermitian form

Pf. Let $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$ be any Hermitian form.

Define $H(v, w) = \displaystyle\int_G \langle g.v, g.w \rangle \, dg$. Then $H(\cdot, \cdot)$ is the required G-invariant Hermitian form. □

(b) If $G \circlearrowright V$ (f.d. repn. / $\mathbb{C}$) and $V' \subset V$ is a subrepn. Then $\exists\, V'' \subset V$ a subrepn. s.t. $V = V' \oplus V''$.
(take $V'' = (V')^{\perp}$ w.r.t. a G-invariant Hermitian form).

(c) Schur's Lemma. V_1, V_2 f.d. irreducible repns. of G. Then

$$\mathrm{Hom}_G(V_1, V_2) = \begin{cases} 0 & \text{if } V_1 \not\simeq V_2 \\ \mathbb{C} & \text{if } V_1 \simeq V_2 \end{cases}$$

Pf. If $\xi : V_1 \longrightarrow V_2$ is a hom. of G-repns. then $\mathrm{Ker}(\xi) \subset V_1$ is a subrepn. $\Rightarrow \begin{cases} \mathrm{Ker}(\xi) = V_1 \Rightarrow \xi = 0 \\ \mathrm{Ker}(\xi) = 0 \Rightarrow V_1 \subset V_2 \text{ is a subrepn. By irr. of } V_2 \\ \quad \xi : V_1 \to V_2 \text{ is iso.} \end{cases}$

In the latter case, let $\lambda \in \mathbb{C}$ be an eigenvalue of ξ. Then $\mathrm{Ker}(\xi - \mathrm{Id} \cdot \lambda) \subset V_1$ is a non-zero subrepn. $\Rightarrow \operatorname{Ker}(\xi - \lambda\cdot \mathrm{Id}) = V \Rightarrow \xi = \lambda\cdot\mathrm{Id}$. $\square$

(d) Schur's orthogonality relations. Let V_1, V_2 be two irr. f.d. repns. of G., with G-invariant Hermitian forms $H_1(\cdot,\cdot)$ and $H_2(\cdot,\cdot)$ resp.

(i) $V_1 \not\cong V_2$. For every $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$ we have

$$\int_G H_1(gu_1, v_1)\,\overline{H_2(gu_2, v_2)}\,dg = 0$$

(ii) If $V_1 = V_2$, $\displaystyle\int_G (gu_1, v_1)\,\overline{(gu_2, v_2)}\,dg = \frac{(u_1, u_2)\,\overline{(v_1, v_2)}}{\dim V}$

Pf for any $\mathbb{C}$-linear map $\xi : V_2 \to V_1$, let $L : V_2 \to V_1$ be defined by $\forall v_2 \in V_2$ $L(v_2) = \displaystyle\int_G g.(\xi(g^{-1}.v_2))\,dg \in V_1$. Then $L : V_2 \to V_1$ is a hom. of repns. of G. Thus $L \equiv 0$ if $V_1 \not\cong V_2$ in which case: $0 = H_1(Lv_2, v_1) = \displaystyle\int_G (g\,\xi\,g^{-1}.v_2, v_1)\,dg$

(take $\xi(v_2') = (v_2', u_2).u_1$)

$$= \int_G (g\,(g^{-1}v_2, u_2)\,u_1,\ v_1)\,dg$$

$$= \int_G (gu_1, v_1)\,(g^{-1}v_2, u_2)\,dg = \int_G (gu_1, v_1)\,\overline{(gu_2, v_2)}\,dg$$

Now, if $V_1 = V_2$, we get $L = \lambda \cdot \mathrm{Id}$ where $\lambda \cdot \dim(V) = \mathrm{Tr}(L) = \mathrm{Tr}(\xi)$. Again take $\xi(w) = (w, u_2) \cdot u_1$ to get

$$\frac{\mathrm{Tr}(\xi)}{\dim V} \overline{(v_1, v_2)} = \frac{(u_1, u_2)}{\dim V} \overline{(v_1, v_2)}$$

$$= \lambda \overline{(v_1, v_2)} = (L v_2, v_1) = \int_G (g \xi g^{-1} \cdot v_2, v_1)\, dg$$

$$= \int_G (g^{-1} v_2, u_2)(g u_1, v_1)\, dg$$

$$= \int_G (g u_1, v_1) \overline{(g u_2, v_2)}\, dg \quad \text{as required} \qquad \square$$