Lecture 22

(22.0) Recall : G = a connected compact Lie group of dimension n.

We fixed a left-invariant n-form ϕ on G, uniquely determined by the requirement that $\int_G 1\,\phi = 1$. We switched back to the standard notation $\int_G f(x)\,dx$ for integral of $f \in C(G)$ (continuous $\mathbb{C}$-valued fns. on G).

- Right invariance (G is unimodular) : $\int_G f(xg)\,dx = \int_G f(x)\,dx$
- $G \circlearrowright V$ f.d. repn. / $\mathbb{C}$ $\rightsquigarrow$ V has a G-invariant Hermitian form
- Every f.d. representation of G is a direct sum of irreducible repns.

Matrix coefficients. For a f.d. representation of G, say V, and $v_1, v_2 \in V$ we have a continuous ($\mathbb{C}$-valued) fn. on G

$$\Phi^V_{v_1,v_2}(g) := (gv_1, v_2) \in \mathbb{C}$$

Schur's orthogonality relations :

(i) $V_1 \not\cong V_2$ irreducible f.d. repns. of G. Then

$$\int_G \Phi^{V_1}_{u_1,v_1}(g)\, \overline{\Phi^{V_2}_{u_2,v_2}(g)}\, dg = 0 \quad \forall \begin{array}{l} u_1, v_1 \in V_1 \\ u_2, v_2 \in V_2 \end{array}$$

(ii) V : irr. f.d. repn. of G

$$\int_G \Phi^V_{u_1,v_1}(g)\, \overline{\Phi^V_{u_2,v_2}(g)}\, dg = \frac{(u_1,u_2)\,\overline{(v_1,v_2)}}{\dim V}$$

(22.1) The pairing $\langle f_1, f_2 \rangle_{L^2} := \int_G f_1(g)\overline{f_2(g)}\, dg$ $(f_1, f_2 : G \to \mathbb{C}$ continuous functions for now) is called (L^2-) inner product. Schur's orthogonality relations imply that:

If $\{V_\lambda\}_{\lambda \in \Lambda}$ is a set of mutually non-isomorphic irreducible f.d. repns of G, $\{v_1(\lambda), \ldots, v_{d_\lambda}(\lambda)\}$ $(d_\lambda = \dim V_\lambda)$ an orthonormal basis of V_λ (w.r.t. a fixed G-invariant Hermitian form on V_λ), then $\Phi^\lambda_{ij} := d_\lambda^{\frac{1}{2}} \Phi^{V_\lambda}_{v_i(\lambda), v_j(\lambda)}$ $(1 \le i, j \le d_\lambda)$

$\{\Phi^\lambda_{ij}\}_{\substack{\lambda \in \Lambda \\ 1 \le i,j \le \lambda}}$ is a set of orthonormal vectors (w.r.t. L^2-inner product).

(22.2) Easy example. $G = S^1$ $(\simeq \mathbb{R}/_{2\pi\mathbb{Z}})$ with invariant form $\frac{d\theta}{2\pi}$. For each $n \in \mathbb{Z}$ we have 1-dim'l irreducible repn given by

$$\begin{array}{ccc} S^1 & \longrightarrow & S^1 = U(1) \subset GL_1(\mathbb{C}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ z & \longmapsto & z^n \end{array}$$

. Schur's orthogonality rel^{n}s become

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta}\, d\theta = \delta_{m,n}.$$

Peter-Weyl Thm: $L^2(S^1)$ has o.n. basis $\{e^{in\theta}\}_{n \in \mathbb{Z}}$.

(Fourier Series of a square-integrable fn.)

(22.3) Some measure theory. (towards defn. of L^p-spaces).

Definition. A measure space is a triple (X, Σ, μ) when X is a set Σ is a collection of subsets of X and $\mu : \Sigma \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ s.t.

- $\phi, X \in \Sigma$ • $A \in \Sigma \Rightarrow X \setminus A \in \Sigma$ • $\{A_i\}_{i=1}^{\infty} \subset \Sigma \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$

(we say Σ is a σ-algebra)

- $\mu(\phi) = 0$ • $\{A_i\}_{i=1}^{\infty} \subset \Sigma$ and $A_i \cap A_j = \phi \ \forall i, j$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

We say μ is <u>finite</u> if $\mu(X) < \infty$; μ is σ-finite if $X = \bigcup_{i=1}^{\infty} X_i$ s.t. $\mu(X_i) < \infty$. A measure space (X, Σ, μ) is said to be <u>complete</u> if

$$\forall E \subset X \text{ s.t. } \inf\{\mu(A) : E \subset A \text{ and } A \in \Sigma\} = 0$$
$$\Rightarrow E \in \Sigma$$

(22.4) Remarks. (i) If μ is finite we may as well assume $\mu(X) = 1$ (Probability measure).

(ii) Any finite (or more generally σ-finite) measure space admits a unique completion. The completion is constructed as follows:

- define $\forall E \subset X$, $\mu^*(E) = \inf\{\mu(A) : E \subset A \text{ and } A \in \Sigma\}$
- a subset $B \subset X$ belongs to $\overline{\Sigma}$ (completion) $\Leftrightarrow$

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c) \quad \forall E \subset X$$

Then we prove that $\overline{\Sigma} \supset \Sigma$ is another σ-algebra and

$\mu(B) := \inf\{\mu(A) : B \subset A,\ A \in \Sigma\}$ is the extension of μ to $\overline{\Sigma}$.
$\forall B \in \overline{\Sigma}$

(iii) In practice, we only define μ on an <u>algebra</u> <u>of subsets</u> of X

$\mathcal{S} \subset 2^X$ s.t.
- $\phi \in \mathcal{S}$
- $A \in \mathcal{S} \Rightarrow X \setminus A \in \mathcal{S}$
- $A_1 \ldots A_n \in \mathcal{S} \Rightarrow A_1 \cup \cdots \cup A_n \in \mathcal{S}$

<u>Caratheodory Extn. Thm.</u> states that if $\mu : \mathcal{S} \to \mathbb{R}_{\geq 0}$ is finite (more generally σ-finite)

and countably additive
$$\begin{cases} \mu(\phi) = 0 \ ; \ \mu(X) < \infty \\ \{A_i\}_{i=1}^{\infty} \subset \mathcal{S} ;\ A_i \cap A_j = \phi \text{ and } \bigcup_{i=1}^{\infty} A_i \in \mathcal{S} \\ \quad \text{then } \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \end{cases}$$

Then μ admits a unique (complete) extn. to σ-algebra generated by $\mathcal{S}$.
The proof of Caratheodory's Thm is exactly same as the one sketched in Remark (ii) above. That is,

$$\forall E \subset X,\ \text{define } \mu^*(E) := \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) : \begin{array}{l} E \subset \bigcup_{i=1}^{\infty} A_i \\ A_i \in \mathcal{S}\ \forall i = 1, \ldots \end{array} \right\}$$

Define $\mathcal{M}(\mu) = \{ B \subset X \text{ s.t. } \mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)\ \forall E \subset X \}$

Prove that $\mathcal{M}(\mu)$ is σ-algebra containing $\mathcal{S}$ and $\mu : \mathcal{M}(\mu) \to \mathbb{R}_{\geq 0}$

extends $\mu : \mathcal{S} \to \mathbb{R}_{\geq 0}$.

~~From now on (X, Σ, μ) will be complete, probability measure space i.e. $\mu(X) = 1$.~~

(22.5) Borel σ-algebra: for a Hausdorff (locally compact) space X, we take Σ = smallest σ-algebra containing all open sets.
(Borel σ-algebra)

e.g. $X = \mathbb{R}$. Let $\mathcal{I} = \{ (a,b] \text{ where } -\infty \leq a < b \leq \infty \}$ $\quad$ $[a,\infty] = (a,\infty)$ convention

$\mathcal{S} = \{ I_1 \sqcup \dots \sqcup I_n : n \geq 1, I_j \in \mathcal{I} \}$ ← algebra of subsets

Borel σ-algebra $= \Sigma_{\mathbb{R}} =$ σ-alg. generated by $\mathcal{S}$ (or $\mathcal{I}$).

(e.g. $(a,b) = \bigcap_{n \geq 1} (a, b + \frac{1}{n}] \in \Sigma$)

Lebesgue Measure $\mu((a,b]) = b-a$ extended to Σ.

⋯> Same for $X = \mathbb{C}$ (σ-algebra generated by open discs or rectangles
μ = area of disc or rectangle).

~~Our case $X = G$~~

~~Σ = σ-algebra generated by compact sets~~ Later!

Observation

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* For $\mathbb{C}$-valued functions we treat real and imaginary parts separately.

(22.6) Let (X, Σ, μ) be a (complete, probability) measure space. A real valued function $f : X \to \mathbb{R}$ is said to be measurable if $f^{-1}(E) \in \Sigma \quad \forall E \in \Sigma_{\mathbb{R}}$

Easy properties: (i) $f_1, \cdots, f_n : X \to \mathbb{R}$ measurable $\Rightarrow \sum_{i=1}^{n} a_i f_i$ is measurable
(countable operations → see page 12 below) $a_1, \cdots, a_n \in \mathbb{R}$

& $\prod_{i=1}^{n} f_i$ is measurable.

(ii) $f : X \to \mathbb{R}$ measurable $\Rightarrow$ $f_+(x) := \mathrm{Max}\{0, f(x)\}$, $f_-(x) := -\mathrm{Min}\{0, f(x)\}$ are measurable

(iii) $\forall\, E \in \Sigma$, $\mathbb{1}_E : X \longrightarrow \mathbb{R}$ (step function) defined by $\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$ is measurable.

<u>Simple functions</u> are finite linear combinations of step functions

<u>Prop. (22.7)</u>. Let $f : X \longrightarrow \mathbb{R}_{\geq 0}$ be a measurable function. Then $\exists\, \{f_i\}_{i=1}^{\infty}$ simple functions (non-negative) s.t.

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \ \& \ \lim_{n \to \infty} f_n(x) = f(x)$$

Proof. Let $I_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ $E_{n,k} = f^{-1}(I_{n,k})$ and define $h_n = \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbb{1}_{E_{n,k}}$. Now $h_n \leq f < h_n + \frac{1}{2^n}$ and $0 \leq h_1 \leq h_2 \leq \cdots \Rightarrow \{h_n\}_{n=1}^{\infty}$ converge uniformly to f.

If f is bounded, each h_n is a simple function. If f is not bounded, we must take $f_n = \min(h_n, n)$ and get pointwise convergence instead of uniform convergence $\square$

(iii) (Fatou's Lemma) f_n $(n=1,2,\dots)$ $: X \to \mathbb{R}_{\geq 0}$ non-negative measurable functions. Then $\displaystyle\int_X (\liminf f_n)\, d\mu \leq \liminf \int_X f_n\, d\mu$

(Defn. A real valued measurable fn. $f: X \to \mathbb{R}$ is said to be <u>integrable</u> if $\displaystyle\int_X f_\pm\, d\mu < \infty$, so $\displaystyle\int_X f\, d\mu := \int_X f_+ d\mu - \int_X f_- d\mu$.

A complex-valued measurable $f: X \to \mathbb{C}$ is integrable if $|f| : X \to \mathbb{R}_{\geq 0}$ is (or equivalently $\mathrm{Re}(f), \mathrm{Im}(f)$ are integrable.

$\mathcal{L}^1(X, \Sigma, \mu) :=$ $\mathbb{C}$-vector space of integrable functions.)

(iv) (Lebasgue's dominated convergence theorem). Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions from $\mathcal{L}^1(X, \Sigma, \mu)$. Assume $\exists\, g: X \to \mathbb{R}_{\geq 0}$ $\displaystyle\int_X g\, d\mu < \infty$ s.t. $|f_n| \leq g$ $\forall\, n \geq 1$. Assume $\forall\, x \in X$, $\displaystyle\lim_{n\to\infty} f_n(x)$ exists $(=: f(x))$. Then $f \in \mathcal{L}^1(X, \Sigma, \mu)$

and
$$\int_X f(x)\, d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x)$$

(22.8) $\int_X d\mu$. For a non-negative measurable $f : X \to \mathbb{R}_{\geq 0}$

$$(*): \int_X f \, d\mu := \lim_{n \to \infty} \int_X f_n \, d\mu$$ where $\{f_n\}$ are simple non-negative functions approximating f (Prop 22.7)

Here for $\{E_i\}_{i=1}^{p} \subset \Sigma$ s.t. $E_i \cap E_j = \phi$, and $a_1 \ldots a_p \in \mathbb{R}$

$$\int_X \left(\sum_{i=1}^{p} a_i \mathbb{1}_{E_i} \right) d\mu := \sum_{i=1}^{p} a_i \, \mu(E_i)$$ (integral of a simple fn.)

For arbitrary $f : X \to \mathbb{R}$ $\quad \int_X f \, d\mu := \int_X f_+ \, d\mu - \int_X f_- \, d\mu$ (assuming both terms on RHS are finite)

For $\mathbb{C}$-valued $f : X \to \mathbb{C}$ $\quad \int_X f \, d\mu = \int_X \mathrm{Re}(f) \, d\mu + i \int_X \mathrm{Im}(f) \, d\mu$

Theorem (i) The definition of $\int_X f \, d\mu$ given in $(*)$ is independent of the sequence of simple functions $\{f_n\}_{n=1}^{\infty}$.

(ii) (Monotone Convergence Theorem) If f_n $(n = 1, 2, \ldots)$, $f : X \to \mathbb{R}_{\geq 0}$ are non-negative measurable functions s.t.

$$\forall x \in X \quad f_1(x) \leq f_2(x) \leq \ldots \qquad \lim_{n \to \infty} f_n(x) = f(x)$$

Then $$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

(22.9) Proof of Theorem (22.8) (i)

Step 1. Let $\{f_n\}_{n=1}^{\infty}$ be pointwise decreasing seq. of simple functions s.t. $\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in X$. Then $\lim_{n\to\infty} \int_X f_n \, d\mu = 0$.

Pf. : Let $A_n := f_n^{-1}((\varepsilon,\infty))$ ($\varepsilon > 0$ fixed). Then each $A_n \in \Sigma$ and $\bigcap_{n=1}^{\infty} A_n = \phi \implies \lim_{n\to\infty} \mu(A_n) \to 0$ (by countable additivity of μ and $\mu(X)=1$)

Fix $M>0$ s.t. $f_1(x) \le M$ $(\forall x \in X)$. Then $\cancel{\forall_n}$ $f_n = f_n \cdot \mathbb{1}_{A_n} + f_n \cdot \mathbb{1}_{A_n^c}$

$\implies f_n \le M \cdot \mathbb{1}_{A_n} + \varepsilon \mathbb{1}_{A_n^c} \implies \int_X f_n \, d\mu \le M \cdot \mu(A_n) + \varepsilon$ $\left(\mu(A_n^c) \le \underbrace{\mu(X)}_{=1}\right)$

Hence $\lim_{n\to\infty} \int f_n \, d\mu \le \varepsilon$.

Step 2. If $f, \{f_n\}_{n=1}^{\infty}$ are simple functions s.t. $f_1(x) \le \dots \to f(x)$ $\forall x \in X$

Then $\int_X f_1 \, d\mu \le \int_X f_2 \, d\mu \le \dots \to \int_X f \, d\mu$

(apply Step 1 to $\{f - f_n\}_{n=1}^{\infty}$)

Final Step. $\{f_n\}_{n=1}^{\infty}$ $\{g_n\}_{n=1}^{\infty}$ (pointwise) increasing seq. of simple fns s.t. $\forall x \in X$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x)$. Then

$$\lim_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu$$

Pf. Set $h_{m,n} = \operatorname{Min}\{f_m, g_n\}$. Then $h_{m,1}(x) \le h_{m,2}(x) \le \dots$ and $\lim_{n\to\infty} h_{m,n}(x) = f_m(x)$. By Step 2, we get

$$\int f_m \, d\mu = \lim_{n\to\infty} \int h_{m,n} \, d\mu \leq \lim_{n\to\infty} \int g_n \, d\mu$$

$\Rightarrow \lim_{m\to\infty} \int f_m \, d\mu \leq \lim_{n\to\infty} \int g_n \, d\mu$. By flipping the role of m, n above, we get the other inequality $\square$

(22.10) Proof of Thm (22.8) (ii)

$f, f_1, f_2, \ldots : X \longrightarrow \mathbb{R}_{\geq 0}$ are measurable and $\forall x \in X$ $f_1(x) \leq f_2(x) \leq \ldots \longrightarrow f(x)$. This clearly implies that

$\int f_n \, d\mu \leq \int f \, d\mu \Rightarrow \lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$. For the converse, we will prove that for any non-negative simple fn g s.t. $g \leq f$, $\lim_{n\to\infty} \int f_n \, d\mu \geq \int g \, d\mu$.

Set $h_n = \mathrm{Min}\{f_n, g\}$ (increasing $\to g$) and bounded.

So, by Prop. (22.7) $\exists$ simple non-negative g_n s.t

$$g_n \leq h_n \leq g_n + \varepsilon/2^n$$

Set $\tilde{g}_n = \mathrm{Max}\{g_1 \cdots g_n\}$. Then $0 \leq \tilde{g}_1 \leq \tilde{g}_2 \leq \cdots$

$\tilde{g}_n \leq h_n \leq \tilde{g}_n + \frac{\varepsilon}{2^n}$, $\quad g(x) = \lim_{n\to\infty} \tilde{g}_n(x)$. By defn (Step 2 of (22.9) above)

$$\int g \, d\mu = \lim_{n\to\infty} \int \tilde{g}_n \, d\mu \leq \lim_{n\to\infty} \int f_n \, d\mu$$

(22.11) Proof of Thm (22.8) (iii). $\{f_n\}_{n=1}^{\infty}$ are non-negative measurable functions. Set $g_n = \inf\{f_m : m \geq n\}$.

By Thm 22.8 (ii) $\displaystyle\int \liminf f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu$

Since $\int g_n \, d\mu \leq \int f_m \, d\mu \quad \forall m \geq n$ we get

$$\int \liminf f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu \leq \lim_{n\to\infty} \left(\inf\left\{ \int f_m d\mu : m \geq n \right\} \right)$$

$$= \liminf \int f_n \, d\mu \qquad \square$$

(22.12) Proof of Thm (22.8) (iv). $f_n \in \mathcal{L}^1(X, \Sigma, \mu)$ $(n = 1, 2, \dots)$

$|f_n| < g$ for some $g : X \to \mathbb{R}_{\geq 0}$ measurable s.t. $\int g \, d\mu < \infty$.

$f(x) := \lim_{n\to\infty} f_n(x)$ (exists). These assumptions imply $|f| \leq g \Rightarrow f \in \mathcal{L}^1(X, \Sigma, \mu)$

Taking real & imaginary parts separately, we may assume f_n's are $\mathbb{R}$-valued.

$\{g \pm f_n\}_{n=1}^{\infty}$ are all non-neg. and by (iii) above: $\left(\lim_{n\to\infty} g \pm f_n = g \pm f\right)$ $\quad \int g d\mu \pm \int f d\mu \leq \liminf \left(\int g d\mu \pm \int f_n d\mu \right)$

$$\Rightarrow \begin{cases} \int f d\mu \leq \liminf \int f_n d\mu \\ -\int f d\mu \leq -\limsup \int f_n d\mu \end{cases} \Rightarrow \limsup \int f_n d\mu \leq \int f d\mu \leq \liminf \int f_n d\mu$$

$\square$

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions $X \to \mathbb{R}$.

- $E = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable

Proof. $E = \{x \in X : \forall k \ \exists N \text{ s.t. } \forall m, n \geq N \ |f_n(x) - f_m(x)| < \frac{1}{k}\}$

$$= \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m=N}^{\infty} \bigcap_{n=N}^{\infty} \{x \in X : |f_n(x) - f_m(x)| < \tfrac{1}{k}\}$$

→ Similar arguments show that (for instance)

- if $\lim_{n\to\infty} f_n(x)$ exists $=: f(x)$, then f is measurable
  $\forall x \in X$

- in general $\liminf_n f_n(x) =: \underline{f}(x)$, $\limsup_n f_n(x) =: \overline{f}(x)$ are measurable

- $$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if exists} \\ 0 & \text{o/w} \end{cases}$$ is measurable