# Lecture 23



(23.0) Recall: $(X,\Sigma,\mu)$ is a complete measure space with $\mu(X)=1$.

We introduced the notion of measurable functions $f: X \to \mathbb{C}$ and integrable functions $\mathcal{L}^1(X,\Sigma,\mu) = \{ \underset{\text{measurable}}{f: X \to \mathbb{C}} \text{ s.t. } \int_X |f|\,d\mu < \infty \}$

(23.1) Definition $\mathcal{L}^p(X,\Sigma,\mu)$ $(1 \le p \le \infty)$ consists of measurable functions $f: X \to \mathbb{C}$ s.t.

- $p=\infty$ : $\exists M>0$ s.t. $|f(x)| \le M$ for $\mu$-a.e. $x \in X$ (i.e. $\mu(\{x : |f(x)|>M\}) = 0$)
- $1 \le p < \infty$ : $\int_X |f|^p \, d\mu < \infty$

$$L^p(X,\Sigma,\mu) := \mathcal{L}^p(X,\Sigma,\mu) \Big/ f \sim g \text{ if } \mu(\{x \in X : f(x) \neq g(x)\}) = 0$$

(we say $f = g$ $\mu$-almost everywhere)

For $f \in L^p(X,\Sigma,\mu)$, define

$$\|f\|_p := \left[ \int_X |f|^p \, d\mu \right]^{\frac{1}{p}} \quad \text{for } p \neq \infty$$

$$\|f\|_\infty := \inf\{ M>0 \text{ s.t. } |f| \le M \ \mu\text{-a.e.}\} \quad \text{for } p=\infty$$

(23.2) Hölder's inequality. Let $1 < p < \infty$ and $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p$ and $g \in L^q$ we have

$$\int |f g| \, d\mu \leq \left[ \int |f|^p \, d\mu \right]^{\frac{1}{p}} \left[ \int |g|^q \, d\mu \right]^{\frac{1}{q}}$$

$$\left( = \|f\|_p \cdot \|g\|_q \right)$$

Proof. The proof relies on the concavity of the graph of $y = \ln(x)$

i.e. $t \ln(a) + (1-t) \ln(b) \leq \ln(ta + (1-t)b) \quad \forall \; a, b > 0, \; 0 < t < 1$

$\Rightarrow \; a^t b^{1-t} \leq ta + (1-t)b$. Set $t = \frac{1}{p}$, $a = \left( \frac{|f|}{\|f\|_p} \right)^p$ and $b = \left( \frac{|g|}{\|g\|_q} \right)^q$ to get

$$\frac{1}{\|f\|_p \|g\|_q} \int |f g| \, d\mu \leq \frac{1}{p \cdot \|f\|_p^p} \int |f|^p \, d\mu + \frac{1}{q \|g\|_q^q} \cdot \int |g|^q \, d\mu$$

$$= \frac{1}{p} + \frac{1}{q} = 1 \qquad \square$$

(23.3) Theorem. $\forall p$, $1 \leq p \leq \infty$, $L^p(X, \Sigma, \mu)$ is (w.r.t. $\|\cdot\|_p$) complete normed vector space (= Banach space).

Proof. $p = \infty$ case is almost obvious (left as an exercise!). 

Assume $1 \leq p < \infty$ and $q$ is such that $\frac{1}{p} + \frac{1}{q} = 1$ $\left(\begin{smallmatrix} q = \infty \text{ if} \\ p = 1 \end{smallmatrix}\right)$

Below we'll use $p = (p-1)q$ $(1 < p < \infty)$.

$$\int |f+g|^p \, d\mu \leq \int |f+g|^{p-1} (|f| + |g|) \, d\mu$$

(thus triangle inequality follows for $p=1$ case. Assume $p>1$ below)

$$\leq \| |f+g|^{p-1} \|_q \, \|f\|_p + \| |f+g|^{p-1} \|_q \, \|g\|_p \quad \text{by Hölder inequality (23.2)}$$

$$= \left[\int |f+g|^p \, d\mu\right]^{\frac{1}{q}} (\|f\|_p + \|g\|_p)$$

$$\Rightarrow \|f+g\|_p = \left[\int |f+g| \, d\mu\right]^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p \quad \text{(Triangle inequality)}$$

(This also proves that $\mathcal{L}^p$ and hence $L^p$ is a $\mathbb{C}$-vector space).

The rest of the axioms of a normed vector space are trivial to verify:

$$\|f\|_p \geq 0 \; ; \; \|f\| = 0 \Leftrightarrow f = 0 \; ; \; \|af\| = |a| \cdot \|f\|$$
$$\forall f \in L^p \; ; \; a \in \mathbb{C}.$$

<u>Completeness</u>: Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $L^p$. For $k \geq 1$

$\exists \, n_k$ s.t. $n, m \geq n_k \Rightarrow \|f_n - f_m\|_p < \frac{1}{2^k}$.

Set $$g_\ell = |f_{n_1}| + \sum_{k=1}^{\ell-1} |f_{n_{k+1}} - f_{n_k}|$$

Note : • $\|g_\ell\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{\ell-1} \|f_{n_{k+1}} - f_{n_k}\|_p$ (By triangle ineq.)

• $g_\ell$'s are non-negative increasing functions.

• $f_{n_\ell} = f_{n_1} + \sum_{k=1}^{\ell-1} f_{n_{k+1}} - f_{n_k} \Rightarrow |f_{n_\ell}| \leq g_\ell \; (\forall \ell \geq 1)$.

Monotone Convergence Theorem ( Thm (22.8) (ii) page 7 of Lecture 22)

$$\int g^p \, d\mu \leq \lim_\ell \|g_\ell\|_p^p$$

$$(g := \sup_\ell g_\ell) \qquad \leq \lim_\ell \left( \|f_{n_1}\|_p + \sum_{k=1}^{\ell-1} \|f_{n_{k+1}} - f_{n_k}\|_p \right)^p$$

$$\leq \left( \|f_{n_1}\|_p + 1 \right)^p$$

$\Rightarrow g \in L^p$. In particular $g$ is finite $\mu$-a.e. and since $|f_{n_k}| \leq g$, we get $\{f_{n_k}(x)\}_k$ converges $\mu$-a.e. Define

$$f(x) = \begin{cases} \lim_{k \to \infty} f_{n_k}(x) & \text{if exists} \\ 0 & \text{o/w} \end{cases}$$

(see page 12 of Lecture 22)

Then $f_{n_k} \to f$ $\mu$-a.e. and $|f_{n_k}| \leq g \in L^p$. By Lebasgue's dominated convergence theorem ( Thm (22.8) (iv) Lecture 22 page 8)

$$\|f_{n_k} - f\|_p \to 0 \quad \text{as } k \to \infty.$$

$\square$

(23.4) Special case $p = q = 2$. $L^2(X)$ has $\langle \cdot, \cdot \rangle$ inner product ⑤

$$\langle f, g \rangle := \int_X f \bar{g} \, d\mu$$

Hölder inequality (23.2) $\Rightarrow$ $|\langle f, g \rangle| \leq \|f\|_2 \cdot \|g\|_2 \quad \forall f, g \in L^2(X)$
(Cauchy. Schwarz)

So $\langle \cdot, \cdot \rangle : L^2(X) \times L^2(X) \to \mathbb{C}$ is defined. Thus we have proved that $L^2(X)$ is a Hilbert space :

Definition. A Hilbert space is a complex vector space, $\mathcal{H}$, together with $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ satisfying

(1) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle \quad \forall \alpha, \beta \in \mathbb{C}, \ x, y, z \in \mathcal{H}$

(2) $\langle y, x \rangle = \overline{\langle x, y \rangle} \quad \forall x, y \in \mathcal{H}$. $\langle x, x \rangle \in \mathbb{R}_{\geq 0}$.

Let $\|x\| := \sqrt{\langle x, x \rangle}$.

(3) $\mathcal{H}$ is a normed vector space, i.e.

$\|\alpha x\| = |\alpha| \cdot \|x\| \qquad \|x + y\| \leq \|x\| + \|y\|$

$\|x\| = 0 \iff x = 0 \qquad (\forall \alpha \in \mathbb{C}; \ x, y \in \mathcal{H})$

(4) $\mathcal{H}$ is complete metric space ($d(x,y) := \|x - y\|$). That is for any $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$, Cauchy seq (i.e. $\forall \varepsilon > 0, \exists N$ s.t. $m, n \geq N \Rightarrow \|x_m - x_n\| < \varepsilon$)

$\exists! \ x \in \mathcal{H}$ s.t. $\lim_{n \to \infty} \|x_n - x\| = 0$.

(23.5) Operators on a Hilbert Space.

$$\mathcal{B}(\mathcal{H}) := \{ f : \mathcal{H} \to \mathcal{H} \text{ s.t. } f \text{ is } \mathbb{C}\text{-linear and continuous} \}$$

$$T \in \mathcal{B}(\mathcal{H}), \ \|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{y \text{ s.t. } \|y\| = 1} \|Ty\|$$

Remark : (i) if $T \in \mathcal{B}(\mathcal{H})$, $\exists\ M > 0$ s.t $\|Tx\| \leq M \cdot \|x\|\ \forall x \in \mathcal{H}$.

(continuous $\Rightarrow$ bounded)

The converse is trivially true.

To see this, assume the contrary. Then for every $n = 1, 2, 3, \ldots$ we should be able to find $x_1, x_2, x_3, \ldots \in \mathcal{H}$ s.t. $\|T x_n\| > n \cdot \|x_n\|$. Let $y_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|}$ so that $\|y_n\| = \frac{1}{\sqrt{n}}$

and $\|T y_n\| = \frac{1}{\sqrt{n}} \frac{\|T x_n\|}{\|x_n\|} > \sqrt{n}$. But $\|y_n\| \to 0$ as $n \to \infty$

so $T$ is not continuous.

(ii) Every Hilbert space admits an orthonormal basis $\{e_i\}_{i \in I}$

That is, $\begin{cases} \langle e_i, e_j \rangle = \delta_{ij}. & \text{(Uses Zorn's Lemma)} \\ x \in \mathcal{H} \text{ can be written uniquely as } x = \sum_{i \in I} x_i e_i \end{cases}$

$$x_i := \langle x, e_i \rangle \in \mathbb{C}$$

$$\|x\|^2 = \sum_{i \in I} |x_i|^2 \quad \text{must then be finite!}$$

(Cor. Unit ball is not compact, if the basis is infinite).

(23.6) $T \in \mathcal{B}(\mathcal{H})$ is said to be

- <u>compact</u> if $\overline{T(\text{Unit Ball})}$ is compact
- <u>Hilbert-Schmidt</u> if $\sum_{i,j \in I} |\langle T e_i, e_j \rangle|^2 < \infty$ for $\{e_i\}_{i \in I}$ an o.n. basis
- <u>Self-adjoint</u> if $T = T^*$ (i.e. $\langle Tx, y \rangle = \langle x, Ty \rangle$)

<u>Prop.</u> (i) If $T$ has finite rank then $T$ is compact.

(ii) $\|T_n - T\| \underset{n \to \infty}{\to} 0$, $T_n$ compact $\Rightarrow T$ is compact.

(iii) If $T$ is Hilbert-Schmidt, then for any two o.n. bases $\{e_i\}$, $\{f_i\}$ ; $\sum_{j,i} |\langle Te_j, e_i\rangle|^2 = \sum_{j,i} |\langle Tf_j, f_i\rangle|^2$

$\left(= \sum_j \|Tf_j\|^2\right.$ see Remark (ii) of previous page)

Hilbert-Schmidt norm

$$\|T\|_{HS}^2 := \sum_{j,i} |\langle Te_j, e_i\rangle|^2$$

(iv) $T$ is Hilbert-Schmidt $\Rightarrow \|T\| \leq \|T\|_{H.S.}$

(v) Hilbert-Schmidt $\Rightarrow$ Compact. (Assume $I$ is countable). (In general, the proof below needs slight modification)

Proof. (i) is trivial.

(ii) : Let $B_1(0)$ = unit ball in $\mathcal{H}$. $\forall x, y \in B_1(0)$ and $n \geq 1$, we have

$$\|Tx - Ty\| \leq 2\|T - T_n\| + \|T_n x - T_n y\| \quad - (*)$$

For $\varepsilon > 0$, pick $N$ s.t. $\|T - T_n\| < \frac{\varepsilon}{4}$ $(\forall n \geq N)$ and $x_1, \dots, x_r \in B_1(0)$ s.t. $\forall y \in B_1(0)$, $\exists j$ with $\|T_N y - T_N x_j\| < \frac{\varepsilon}{2}$. Then we get that (from $(*)$ with $n = N$) $T(B_1(0))$ is covered by $B_\varepsilon(T(x_j))$ $(j = 1 \dots r)$.

(iii) $$\sum_{j,i} |\langle Te_j, e_i\rangle|^2 = \sum_j \|Te_j\|^2 = \sum_{j,i} |\langle Te_j, f_i\rangle|^2$$

$$= \sum_{j,i} |\langle e_j, T^* f_i\rangle|^2 = \sum_i \|T^* f_i\|^2 \quad \text{(independent of } \{e_i\})$$

(iv) Let $\{e_i\}_{i\in I}$ be o.n. basis of $\mathcal{H}$ and $x = \sum_{j\in I} c_j e_j \in \mathcal{H}$

$$\|Tx\|^2 = \sum_i \left|\sum_j c_j \langle Te_j, e_i\rangle\right|^2 \leq \sum_i \left(\sum_j |c_j|^2\right)\left(\sum_j |\langle Te_j, e_i\rangle|^2\right)$$

(Cauchy-Schwarz)

$$= \|x\|^2 \cdot \|T\|_{HS}^2$$

(V) Let $\{e_i\}_{i\in I}$ be o.n. basis of $\mathcal{H}$. $\forall\, n \geq \phi$, define

$$T_n e_i = \begin{cases} Te_i & i \leq n \\ 0 & o/w \end{cases} \quad . \quad (\text{We are assuming } I = \{1, 2, 3, \dots\}$$

$T_n$ has finite rank, hence compact. $T - T_n$ is Hilbert-Schmidt and $\|T - T_n\|_{HS} \to 0$ as $n \to \infty$ $\Rightarrow$ $\|T - T_n\| \to 0$ as $n \to \infty$ (by (iv))

By (ii) $T$ is compact.

(23.7) **Prop.** Let $T \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator.

(i) If $W \subset \mathcal{H}$ is $T$-invariant subspace (i.e. $T(W) \subset W$) then so is $W^{\perp}$.

(ii) $\forall\, x \in \mathcal{H}$, $\langle Tx, x\rangle \in \mathbb{R}$ (in particular eigenvalues of $T$, if exist, are real)

(iii) $\|T\| = \sup\limits_{\|x\| \leq 1} |\langle Tx, x\rangle|$

(iv) Let $\lambda \neq \mu \in \mathbb{R}$ and $E_\lambda, E_\mu$ be eigenspaces of $T$, of eigenvalue $\lambda$ & $\mu$. Then $E_\lambda \perp E_\mu$

**Proof.** Only (iii) is non-trivial. We first show that for general $T \in \mathcal{B}(\mathcal{H})$, $\|T\| = \sup\limits_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |\langle Tx, y\rangle|$. By Cauchy-Schwarz $|\langle Tx, y\rangle| \leq \|T\|$ $\forall\, x, y \in \mathcal{H}$ s.t. $\|x\|, \|y\| \leq 1$

Conversely, if $T \neq 0$, for any $y \in \mathcal{H}$ s.t. $Ty \neq 0$, take $x = \dfrac{Ty}{\|Ty\|}$ to get $\langle Ty, x\rangle = \|Ty\|$. Taking supremum over $y \in \mathcal{H}$ s.t. $\|y\| = 1$ gives the desired inequality.

Now if $T = T^*$, let $\alpha = \sup\limits_{\|x\| \leq 1} |\langle Tx, x\rangle|$. Clearly $\alpha \leq \|T\|$.

Conversely it is enough to prove that $|\langle Tx, y\rangle| \leq \alpha \|x\| \cdot \|y\|$ $\forall\, x, y \in \mathcal{H}$.

Multiplying $y$ by a complex number of modulus 1, if necessary, we can assume that $\langle Tx, y\rangle \in \mathbb{R}$. Using $T = T^*$ we get

$$\langle T(x \pm y), x \pm y\rangle = \langle Tx, x\rangle \pm 2\langle Tx, y\rangle + \langle Ty, y\rangle$$

$$\Rightarrow \quad 4\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle$$

$$\Rightarrow \quad |\langle Tx, y\rangle| \leq \frac{\alpha}{4}\left(\|x+y\|^2 + \|x-y\|^2\right)$$

$$\leq \frac{\alpha}{2}\left(\|x\|^2 + \|y\|^2\right)$$

Assuming $x \neq 0$, $y \neq 0$, we apply this inequality to $\sqrt{\frac{\|y\|}{\|x\|}} \cdot x$ and $\sqrt{\frac{\|x\|}{\|y\|}} \cdot y$ to get $|\langle Tx, y\rangle| \leq \alpha \|x\| \cdot \|y\|$ as desired. $\square$

(23.8) Spectral Theorem. Let $T$ be a self-adjoint compact operator. Then $\mathcal{H}$ has an orthonormal basis consisting of (real e.v.'s) eigenvectors of $T$. Moreover $\forall \lambda \in \mathbb{R}$, $\lambda \neq 0$, $\dim \mathcal{H}_\lambda < \infty$, and $\forall \varepsilon > 0$ $\{\lambda : |\lambda| > \varepsilon \text{ and } E_\lambda \neq 0\}$ is a finite set.

Proof. Assume $\mathcal{H} \neq (0)$ and $T \neq 0$. By Prop (23.7) (iii) above, we can find $x_n \in \mathcal{H}$ with $\|x_n\| = 1$, $\lambda = \lim_{n\to\infty} \langle Tx_n, x_n\rangle$ s.t. $|\lambda| = \|T\|$.

$T$: compact $\Rightarrow$ (taking subsequence if necessary) $Tx_n \to y \in \mathcal{H}$. ($y \neq 0$ since $\lambda \neq 0$)

Now $$\|Tx_n - \lambda x_n\|^2 = \|Tx_n\|^2 - 2\lambda\langle Tx_n, x_n\rangle + \lambda^2\|x_n\|^2$$
$$\leq 2\|T\|^2 - 2\lambda\langle Tx_n, x_n\rangle$$

$\Rightarrow \|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$.

Now $Tx_n \to y$, so we must have $\lambda x_n \to y$

$\Rightarrow \lambda Tx_n \to Ty$. Combining, we get $Ty = \lambda y$.

Apply Zorn's lemma to get a max'l collection of o.n. set of eigenvectors of $T$, say $\{e_i\}_{i \in I}$. Let $W \subset \mathcal{H}$ be the closure of the span of $\{e_i\}_{i\in I}$. If $W^\perp \neq (0)$, we apply the previous argument to $T|_{W^\perp}$ ($W$ is $T$-invariant $\Rightarrow$ so is $W^\perp$ Prop (23.7) (i)), to get an eigenvector of $T$ in $W^\perp$, contradicting maximality of $\{e_i\}$.

For the rest, note that if $\lambda_i \in \mathbb{R}$ is the eigenvalue of $e_i$, then

$$\|Te_i - Te_j\|^2 = |\lambda_i|^2 + |\lambda_j|^2$$

$e_i \in B_1(0) \subset \mathcal{H}$; i.e. $\|e_i\|^2 = 1$

- If $\dim E_\lambda = \infty$ for some $\lambda \neq 0$, we get infinitely many $e_i$'s in $B_1(0)$ s.t. $\|Te_i - Te_j\|^2 = 2\lambda^2$ contradicting compactness.

- If for some $\varepsilon > 0$, the set $S := \{i : |\lambda_i| > \varepsilon\}$ is infinite, we get $\forall i,j \in S$ : $\|Te_i - Te_j\| > 2\varepsilon^2$ again contradicting compactness. $\square$

(23.9) Examples of Hilbert-Schmidt operators. Let $(X, \Sigma, \mu)$ be a measure space as before. Let $K \in L^2(X \times X)$. Define

$$(T_K f)(x) = \int_X K(x,y)\, f(y)\, d\mu_y$$

Prop. $T_K \in \mathcal{B}(L^2(X, \Sigma, \mu))$ is a Hilbert-Schmidt operator. $\|T_K\|_{H.S.} = \|K\|_{L^2(X\times X)}$

Proof. Let $K_x : y \longmapsto K(x,y) \in L^2(X, \Sigma, \mu)$ $\forall x \in X$

(Fubini's Thm for product of measure spaces).

Let $\{e_i\}_{i \in I}$ be an o.n. basis of $L^2(X)$.

$$\|T_K\|_{H.S.} = \sum_{i \in I} \|T_K e_i\|^2 = \sum_{i \in I} \int_X |(T_K e_i)(x)|^2 \, d\mu_x$$

$$= \sum_{i \in I} \int_X |\langle K_x, \bar{e}_i \rangle|^2 \, d\mu_x = \sum_{i \in I} \int_X |\langle \bar{K}_x, e_i \rangle|^2 \, d\mu_x$$

$$= \int_X \|\bar{K}_x\|^2_{L^2(X)} \, d\mu_x = \|K\|^2_{L^2(X \times X)} \qquad \square$$