Lecture 24

(24.0) G = connected compact Lie group. $\int_G f(x)\,dx$ left (& right) invariant integration on G s.t. $\int_G 1\,dx = 1$ (see Lecture 21).

[Riesz Representation Thm (see Lecture 25 ~~below~~)] $\Rightarrow$ we have an invariant measure μ on Σ_G = Borel σ-algebra.

$$L^2(G) = L^2(G, \Sigma_G, \mu) = \left\{ f : G \to \mathbb{C} \text{ measurable s.t. } \int_G |f|^2 d\mu < \infty \right\}$$

(24.1) We have a G-action on $L^2(G)$ by

$$(\sigma \cdot f)(x) = f(\sigma^{-1}x) \quad \forall f \in L^2(G),\ \sigma, x \in G.$$

Recall: for a f.d. $G \circlearrowright V$, and $(\cdot,\cdot) : V \times V \to \mathbb{C}$ invariant Hermitian form, we defined matrix coeff $\forall v, w \in V$

$$\Phi^V_{v,w}(g) = (g \cdot v, w)$$

Prop. (i) G action on $L^2(G)$ preserves $\langle \cdot, \cdot \rangle_{L^2}$.

(ii) For a fixed $v \in V$, the map $\begin{matrix} V & \longrightarrow & L^2(G) \\ w & \longmapsto & \Phi^V_{v,w} \end{matrix}$ is G-hom.

(iii) $\Phi^V_{v,w}(g^{-1}) = \overline{\Phi^V_{w,v}(g)} = \Phi^{V^*}_{\ell_w, \ell_v}(g)$

where for $v \in V$, $\ell_v \in V^*$ is defined by $\ell_v(u) = \langle u, v \rangle$

Note $v \longmapsto \ell_v$ is not $\mathbb{C}$-linear.

Proof. (i) $\langle \sigma\cdot f_1, \sigma\cdot f_2\rangle_{L^2} = \int_G (\sigma\cdot f_1)(x)\,\overline{(\sigma\cdot f_2)(x)}\,d\mu_x$

$\forall \sigma\in G$, $f_1, f_2 \in L^2(G)$

$$= \int_G f_1(\sigma^{-1}x)\,\overline{f_2(\sigma^{-1}x)}\,d\mu_x \underset{\text{by invariance of } \mu}{=} \int_G f_1(x)\,\overline{f_2(x)}\,d\mu_x$$

$$= \langle f_1, f_2\rangle_{L^2}$$

(In particular, $\|\sigma\cdot f\| = \|f\| \Rightarrow$ each $\sigma \circlearrowright L^2(G)$ is bounded (unitary) operator)

$\sigma^* = \sigma^{-1} \rightarrow$ (unitary) operator

(ii) $\sigma\cdot\Phi^V_{v,w}(x) = \langle \sigma^{-1}x\cdot v, w\rangle = \langle x\cdot v, \sigma\cdot w\rangle$

$$= \Phi^V_{v,\sigma w}(x)$$

(iii) $\Phi^V_{v,w}(g^{-1}) = \langle g^{-1}v, w\rangle = \langle v, gw\rangle = \overline{\langle gw, v\rangle}$

$$= \overline{\Phi^V_{w,v}(g)}$$

. Last part is similar.

(24.2) Cor. Let $H \subset L^2(G)$ be a f.d. G-invariant subspace.

Then $H \subset$ Span of Matrix Coefficients.

Pf. Let $\{h_1, \ldots, h_N\}$ be an o.n. basis of $\mathcal{H}$. Then $\forall g\in G$

$$g\cdot h_i = \sum_{j=1}^{N} \Phi^{\mathcal{H}}_{h_i,h_j}(g)\, h_j$$

$\rightsquigarrow$ evaluate at e

$$h_i(g^{-1}) = \sum \underbrace{\Phi^{\mathcal{H}}_{h_i,h_j}(g)}_{\text{Matrix Coeff of } \mathcal{H}^* \text{ from (iii) of Prop (24.1)}}\ \overbrace{h_j(e)}^{\text{scalars}}$$

$\square$

(24.3) Since G is second countable (cf Homework 1, Problem 4 (c))

$L^2(G)$ is separable (i.e. has countable o.n. bases).

Now for a collection of mutually non-iso irreducible f.d. repns $\{V_\lambda\}_{\lambda\in\Lambda}$, let $d_\lambda = \dim V_\lambda$ and let $\{v_i(\lambda)\}_{i=1\ldots d_\lambda}$ be an o.n. basis of V_λ. By Schur's orthogonality rel^{n}s

$$\left\{ \Phi^{\lambda}_{i,j} := d_\lambda^{\frac{1}{2}}\, \Phi^{V_\lambda}_{v_i(\lambda),\, v_j(\lambda)} \right\}$$

form an orthonormal set of vectors in $L^2(G)$

Cor. The set of iso. classes of irr. f.d. repns of G is at most countable.

(24.4) Thus we have

$$\bigoplus_{\lambda\in\Lambda} V_\lambda^{\oplus d_\lambda} \hookrightarrow L^2(G).$$

$$v \in i^{th} \text{ copy of } V_\lambda \longrightarrow \phi^{V_\lambda}_{v_i(\lambda),\, v}$$

Let $\mathcal{H}$ = closure of the span of matrix coefficients in $L^2(G)$.

[Peter-Weyl] $\mathcal{H} = L^2(G)$

Pf. If not, $\exists\ \phi \in \mathcal{H}^{\perp}$, $\phi \neq 0$. Use ϕ to define

$$K(x,y) = \cancel{\phi(x^{-1}y)} \in L^2(G\times G) \qquad K(x,y) = \phi(x^{-1}y)$$

(Hilbert-Schmidt (23.9) page 10)

and thus $T \in \mathcal{B}(L^2(G))$

$$\cancel{(Tf)(y) = \int_G \phi(x^{-1}y) f(y)\, d\mu_y}$$

$$(Tf)(x) := \int_G \phi(x^{-1}y)\, f(y)\, d\mu_y$$

Claim T & T^* commute with $G \circlearrowright L^2(G)$

(say $\sigma \in G \longmapsto \pi(\sigma) \in \mathcal{B}(L^2(G))$)

Pf. $$(T(g\cdot f))(y) = \int_G \phi(y^{-1}\sigma) f(g^{-1}\sigma)\, d\mu_\sigma$$

$$= \int_G \phi(y^{-1}g\sigma) f(\sigma)\, d\mu_\sigma = \int_G \phi((g^{-1}y)^{-1}\sigma) f(\sigma)\, d\mu_\sigma$$

$$= (Tf)(g^{-1}y) = (g\cdot(Tf))(y).$$

Now $\pi(g^{-1}) = \pi(g)^*$. Using this, we get

$$\pi(g)\, T^* = (T\, \pi(g^{-1}))^* = (\pi(g^{-1})\, T)^* = T^* \pi(g) \text{ as claimed.}$$

Apply spectral theorem (Thm (23.8) page 9) to T^*T

$$L^2 G = \operatorname{Ker}(T^*T) \oplus \bigoplus_{\lambda \neq 0} E_\lambda \xrightarrow[\text{(projection)}]{P} \mathcal{H}^\perp$$

Each E_λ is f.d. G-invariant $\Rightarrow E_\lambda \subset \mathcal{H}$ (see Cor (24.2)).

This implies $\phi \in \mathcal{H}^\perp \subset \operatorname{Ker}(T^*T) = \operatorname{Ker}(T)$ (since $T^*Tx = 0$

$\Rightarrow \langle T^*Tx, x\rangle = 0$

$\Rightarrow \|Tx\|^2 = 0 \Rightarrow Tx = 0$)

So $0 = (T\bar{\phi})(e) = \int_G |\phi(y)|^2 d\mu_y = \|\phi\|$

$\Rightarrow \phi = 0$ contradiction

$\square$