Lecture 25

(25.0) Recall: G is a connected compact Lie group. In Lectures 20, 21 we proved the existence of an invariant integration $\int_G f(x)\,dx$, normalized to have $\int_G 1\,dx = 1$.

An instance of Riesz Representation Theorem: we have a left (and right) invariant measure μ on the Borel σ-algebra Σ_G (smallest σ-algebra containing all open sets) called Haar measure.

Idea: define $\mu(\mathcal{U}) = \sup \left\{ \int_G f(x)\,dx \;:\; \begin{array}{l} f \in C(G) \text{ real-valued cnts.} \\ 0 \le f \le 1 \\ \mathrm{Supp}(f) \subset \mathcal{U} \end{array} \right\}$

(25.1) Last time we proved Peter-Weyl Theorem

$$L^2(G) = \widehat{\bigoplus_{\lambda \in \Lambda}} V_\lambda^{\oplus d_\lambda}$$

as repns. of G

Λ = set of iso.-classes of f.d. repns. (irred.) of G

$d_\lambda = \dim V_\lambda$

Cor. G admits a faithful finite-dimensional representation. (i.e. $\rho : G \to GL(V)$ s.t. ρ is injective). In particular, $G \hookrightarrow U(N)$ $(N = \dim V)$ as a closed subgroup.

Proof. We claim that for any $g \neq e$, $g \in G$, there exists a f.d. repn., say $\rho^{(g)} : G \to GL(V^{(g)})$ s.t. $\rho^{(g)}(g) \neq \mathrm{Id}$

Since on the contrary, $\Phi^V_{v,w}(g) = \Phi^V_{v,w}(e)$ $\forall$ f.d. repn V & $v, w \in V$, and by Peter-Weyl, $f(g) = f(e)$ $\forall$ $f \in L^2(G)$. But we can take an open set $\mathcal{U} \subset G$ containing g s.t. $e \notin \mathcal{U}$ and $\mathbb{1}_{\mathcal{U}} \in L^2(G)$ satisfies $\mathbb{1}_{\mathcal{U}}(g) = 1$, $\mathbb{1}_{\mathcal{U}}(e) = 0$.

Furthermore, $\mathrm{Ker}(\rho^{(g)}) \subset G$ must have strictly smaller dim. Since, otherwise it will be a subgroup containing an open nbd. of e, implying that $\mathrm{Ker}(\rho^{(g)}) = G$ which will contradict $\rho^{(g)}(g) \neq \mathrm{Id}$.

Now start from $g_0 \in G$ $(g_0 \neq e)$ and let G_1 = identity component of $\mathrm{Ker}(\rho^{(g_0)})$. If $G_1 \neq \{e\}$, let $g_1 \in G_1$ $(g_1 \neq e)$ and take

$$G_2 = \mathrm{Ker}(\rho^{(g_0)} \oplus \rho^{(g_1)}).$$

Since dimension strictly drops at each step, we will find a repn (f.d.) $\rho_1 : G \to GL(V_1)$ s.t. $\mathrm{Ker}(\rho_1)$ is discrete, hence by compactness of G, finite, say $\mathrm{Ker}(\rho_1) = \{h_1, \dots, h_\ell\} \cup \{e\}$. Take $\rho = \rho_1 \oplus \bigoplus_{j=1}^{\ell} \rho^{(h_\ell)}$ $\square$

(25.2) Application to structure theory

Recall Thm (13.5) Lecture 13 page 3 - every $\mathbb{C}$-semisimple Lie algebra admits a unique compact real form (i.e. a real Lie subalg. $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ s.t. Killing form is negative-definite on $\mathfrak{g}$).

3

Prop. Let G be a (connected) compact Lie group. $\mathfrak{g} = \mathrm{Lie}(G)$

Then $\mathfrak{g}$ is reductive (i.e. $\mathfrak{g} = \underbrace{\mathfrak{z}}_{\substack{\text{abelian} \\ (\text{center of } \mathfrak{g})}} \oplus \underbrace{[\mathfrak{g},\mathfrak{g}]}_{\text{semisimple}}$).

Killing form on $\mathfrak{g}$ is negative semidefinite ($\mathfrak{z}$ = radical of $K(\cdot,\cdot)$)

K is negative definite on $\mathfrak{g}_{s.s.} = [\mathfrak{g},\mathfrak{g}]$.

Proof. Pick an inner product on f.d. real vector space $\mathfrak{g}$, say $(\cdot,\cdot)$ and average over G, to make it invariant, i.e.

$$B(x,y) := \int_G (\mathrm{Ad}(g)\cdot x, \mathrm{Ad}(g)\cdot y)\, dg$$

Then we get

(1) $\mathfrak{g}$ = direct sum of G-invariant subspaces (= ideals of $\mathfrak{g}$)

$\mathfrak{z}$ = $\oplus$ of 1-dim'l ideals (min'l)

$[\mathfrak{g},\mathfrak{g}] = \mathfrak{z}^{\perp}$ semisimple (= direct sum of simples).

(2) Each $\mathrm{ad}(X) \circlearrowright \mathfrak{g}$ is skew-symmetric w.r.t. $B(\cdot,\cdot)$

$\Rightarrow$ eigenvalues of $\mathrm{ad}(X)^2 \leq 0$

So $K(x,x) \leq 0 \quad \forall x \in \mathfrak{g}$.

By Cartan's criterion of semisimplicity $K|_{\mathfrak{g}_{ss}}$ is non-deg.

hence $K(X,X) < 0 \quad \forall x \in \mathfrak{g}_{s.s.}$ $\square$

(25.3) Conversely, assume that $\mathfrak{g}$ is real Lie alg. with negative definite Killing form. Let $\mathrm{Aut}^\circ(\mathfrak{g})$ = connected component (of identity) of the group of Lie alg. auto. of $\mathfrak{g}$.

Cor. $\mathrm{Aut}^\circ(\mathfrak{g})$ is compact

Since $\mathrm{Aut}^\circ(\mathfrak{g}) \underset{\text{closed}}{\subset} GL(\mathfrak{g})$ acts on $\mathfrak{g}$ via orthogonal transformations relative to $-K(\cdot,\cdot)$: inner product on $\mathfrak{g}$. Thus $\mathrm{Aut}^\circ(\mathfrak{g})$ is a closed subgroup of $O(\mathfrak{g})$, hence cpct. $\square$

Let G be a connected compact Lie group. $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{z} \oplus \mathfrak{g}_{s.s.}$

(25.4) Thm. If $Z \subset G$ is center of G and $G_{ss} \hookrightarrow G$ Lie subgp. corr. to subalg. $\mathfrak{g}_{ss} \subset \mathfrak{g}$, then

- G_{ss} has finite center
- Z_0 and G_{ss} are closed subgps. of G, and $G = Z_0 \cdot G_{ss}$.

(Z_0 = component of Z containing e).

Proof. Let $\tilde{Z}_0$, $\tilde{G}_{ss}$ be universal covers of Z_0 and G_{ss} resp. Then $\tilde{G} = \tilde{Z}_0 \times \tilde{G}_{ss}$ is simply connected Lie gp w/ $\mathrm{Lie}(\tilde{G}) = \mathfrak{z} \oplus \mathfrak{g}_{ss} = \mathfrak{g}$

$\Rightarrow$ we have a covering map

$$\begin{array}{ccccc} \tilde{G} & = & \tilde{Z}_0 & \times & \tilde{G}_{ss} \\ \downarrow & & & \downarrow & \\ G & \leftarrow & Z_0 & \times & G_{ss} \end{array}$$

$\Rightarrow G = Z_0 \cdot G_{ss}$

<u>G_{ss} has finite center</u>: Let $G \circlearrowright V$ be a faithful f.d. repr.

If $V = V_1 \oplus \dots \oplus V_\ell$ (each $V_1, \dots, V_\ell$ irreducible). By Schur's lemma every $x \in Z_{ss} \subset G_{ss}$ acts as a scalar on V_j.

Note $G_{ss} \xrightarrow{\rho_j} GL(V_j) \xrightarrow{\det} \mathbb{C}^\times$ is 1-dim'l repn. of semisimple group, hence trivial.

$\Rightarrow$ $\rho_j(x)$ acts by scalar s.t. $\rho_j(x)^{\dim V_j} = 1$.

So Z_{ss} has size at most $\prod_{j=1}^{\ell} \dim V_j$ (by faithfulness of $G \circlearrowright V$).

Finally $\mathrm{Aut}^0(\mathfrak{g}_{ss})$ is cpct and $G_{ss} \xrightarrow[\eta]{} \mathrm{Aut}^0(\mathfrak{g}_{ss})$.

$\mathrm{Ker}(\eta) \subset \underset{\uparrow \text{ finite}}{\text{Center of } G_{ss}}$ (being discrete normal subgp - HW2, problem 2)

$\Rightarrow$ G_{ss} is cpct, hence closed in G. $\square$