Lecture 26

(26.0) Recall: G is a connected compact Lie group.

$\mathfrak{g} = \mathrm{Lie}(G)$. Last time we proved:

(1) $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_{s.s.}$ where $\mathfrak{z}$ = center of $\mathfrak{g}$ (abelian ideal)

$\mathfrak{g}_{s.s.} = [\mathfrak{g}, \mathfrak{g}]$ is semisimple.

(2) Killing form on $\mathfrak{g}$ is negative semidefinite, $\mathfrak{z}$ = radical of the Killing form & $K\big|_{\mathfrak{g}_{s.s.}}$ is negative definite.

(3) Conversely, if $\mathfrak{a}$ is a real Lie algebra with negative definite Killing form, then $\mathrm{Aut}^0(\mathfrak{a})$ is compact

(recall $\mathrm{Aut}(\mathfrak{a})$ = group of Lie algebra automorphisms of $\mathfrak{a}$

$\cup$

$\mathrm{Aut}^0(\mathfrak{a})$ = connected component of $\mathrm{Id}_{\mathfrak{a}}$.)

(4) Let Z = Center of G, Z^0 = connected component of Z containing $e \in G$ (unit). $G_{ss} \hookrightarrow G$ Lie subgp. corr. to $\mathfrak{g}_{ss} \subset \mathfrak{g}$. Then Z^0 and G_{ss} are closed in G, center of $G_{s.s.}$ is finite, and $G = Z^0 \cdot G_{ss}$.

(26.1) Maximal Torus $T \subset G$ is connected abelian subgroup max'l w.r.t. inclusion. (since for any abelian subgroup A, $\overline{A}$ is again abelian, maximality implies closed).

Prop. $\left\{\begin{array}{l}\text{Maximal tori}\\ \text{in } G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Max'l abelian}\\ \text{subalgebras of } \mathfrak{g}\end{array}\right\}$

Proof. $A \subset G$ a max'l torus. Then $\mathfrak{a} \subset \mathfrak{g}$ is an abelian subalgebra. If $\mathfrak{a}$ is not max'l, say $\mathfrak{a} \subset \mathfrak{a}'$, then $A \subset \overline{A'}$ will contradict maximality of A. Conversely, if $\mathfrak{a} \subset \mathfrak{g}$ is a max'l abelian subalgebra, let $A \hookrightarrow G$ be Lie subgroup corr. to $\mathfrak{a}$ and $\overline{A} \subset G$. Then $\overline{A}$ is max'l torus in G. If $A \subsetneq \overline{A}$, then $\mathrm{Lie}(\overline{A}) \supsetneq \mathfrak{a}$ will contradict maximality. So A is closed and a max'l torus. □

(26.2) Notations : $T \subset G$ maximal torus

$$\mathfrak{g}_0 = \mathrm{Lie}(G) \qquad \mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$$

$$\mathfrak{g} = \underbrace{\mathfrak{z} \oplus \mathfrak{h}'}_{\substack{\text{Cartan subalg.}\\ \mathfrak{h} \text{ of } \mathfrak{g}}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

(root space decomposition of $\mathbb{C}$-semisimple $[\mathfrak{g},\mathfrak{g}]$)

$$\cup$$

$$\mathfrak{g}_0 = \mathfrak{z}_0 \oplus [\mathfrak{g}_0, \mathfrak{g}_0] \qquad \text{Compact real form}$$

so $\mathfrak{t}_0 = \mathrm{Lie}(T) = \mathfrak{z}_0 + i\mathfrak{h}'_{\mathbb{R}}$ $\quad$ ($\mathfrak{h}'_{\mathbb{R}} = \mathbb{R}$- span of h_α $(\alpha \in R)$)

i.e. $\alpha \in R$ takes purely imaginary values on $\mathfrak{t}_0$ and

$T \circlearrowright \mathfrak{g}_\alpha$ is given by $\mathrm{Ad}(t)\cdot \mathfrak{g}_\alpha = \xi_\alpha(t) \cdot \mathfrak{g}_\alpha$

$\xi_\alpha : T \longrightarrow S^1 \subset \mathbb{C}$ multiplicative character. For $X \in \mathfrak{t}_0$

$$\xi_\alpha(\exp(X)) = e^{\alpha(X)} \in S^1 \quad \text{since} \quad \alpha(X) \text{ is purely imaginary.}$$

Moreover, for any $x \in i\left(\mathfrak{h}'_{\mathbb{R}} \setminus \bigcup_{\alpha \in R} \operatorname{Ker}(\alpha)\right) \subset \mathfrak{t}_0$, we have

Centralizer of x in $\mathfrak{g}_0 = \mathfrak{t}_0$.

Prop. Let $\mathfrak{t}_0$ and $\mathfrak{t}'_0$ be two max'l abelian subalgebras of $\mathfrak{g}_0$. Then there exists $g \in G$ s.t. $\operatorname{Ad}(g).\mathfrak{t}_0 = \mathfrak{t}'_0$.

Proof. Let $x \in \mathfrak{t}_0$, $y \in \mathfrak{t}'_0$ be such that $Z_{\mathfrak{g}_0}(x) = \mathfrak{t}_0$, $Z_{\mathfrak{g}_0}(y) = \mathfrak{t}'_0$. Consider the real-valued fn. on G: $g \in G \longmapsto -K(\operatorname{Ad}(g).x, y)$

Let $g_0 \in G$ be such that this function is minimized at g_0.

Then $\forall z \in \mathfrak{g}_0$, $\psi_z : \mathbb{R} \longrightarrow \mathbb{R}$

$$\psi(r) = -K(\operatorname{Ad}(\exp(rz))\operatorname{Ad}(g_0)\cdot x, y)$$

has min. at $r = 0$. Thus $\psi'(0) = 0 \Rightarrow$

$$0 = -K([z, \operatorname{Ad}(g_0)x], y) = -K(z, [\operatorname{Ad}(g_0)x, y])$$

$$\Rightarrow [\operatorname{Ad}(g_0)x, y] \in [\mathfrak{g}_0, \mathfrak{g}_0] \cap \text{Radical of } K = 0$$

so $y \in$ Centralizer of $\operatorname{Ad}(g_0).x = \operatorname{Ad}(g_0).\mathfrak{t}_0$

$\Rightarrow \mathfrak{t}'_0 \subset \operatorname{Ad}(g_0).\mathfrak{t}_0$. By maximality they must be equal. $\square$

Cor. Any two maximal tori of G are conjugate to each other.

(26.3) Theorem. For any $g \in G$, there exists $h \in G$ s.t. $hgh^{-1} \in T$.

Proof. By induction on $\dim(G)$. Base case is trivial.

Notation: $T^y = \{ yty^{-1} : t \in T \}$ $\quad T^G = \bigcup_{y \in G} T^y$

(so that we want to prove $G = T^G$).

Now, by induction hypothesis and $G = Z^\circ \cdot G_{s.s.}$, we will restrict our attention to the case when G is semisimple.

In this case Z_G = center of G is finite and $\dim(G) \geq 3$.

Define $G^\times = G \setminus Z_G \qquad T^\times = T \setminus (T \cap Z_G)$.

<u>Claim</u>. $(T^\times)^G \subset G^\times$ is open and closed.

Note that $G^\times \subset G$ is connected, open dense subset. So the claim implies $(T^\times)^G = G^\times \subset T^G$. But T^G = Image of

$$\left\{ \begin{array}{ccc} G \times T & \longrightarrow & G \\ (g,t) & \longmapsto & gtg^{-1} \end{array} \right\}$$

is closed since $G \times T$ is cpct. Hence

$G = \overline{G^\times} \subset T^G \Rightarrow G = T^G$ as desired.

(26.4) Proof of the claim: I. $(T^\times)^G \subset G^\times$ is closed.

Consider a sequence in $(T^\times)^G$ converging to a point in $G^\times$:

$x_n t_n x_n^{-1} \longrightarrow g \in G^\times$. Here $\{t_n\} \subset T^\times$, $\{x_n\} \subset G$.

By compactness of T and G we may assume (upon taking a subsequence if necessary) that $\begin{array}{l} x_n \longrightarrow x \in G \\ t_n \longrightarrow t \in T \end{array}$, hence $x t x^{-1} = g \in G^\times$.

It remains to note that $t \in T^\times$. Since otherwise $t \in T \cap Z_G$ implying $g = x t x^{-1} = t \in Z_G$ contradicting the fact that $g \in G^\times = G \setminus Z_G$.

(26.5) Proof of the claim : II. $(T^\times)^G \subset G^\times$ is open.

Fix $t \in T^\times = T \setminus (T \cap Z_G)$. Let $H = (Z_G(t))^\circ_\bullet$ (connected component of e)

Then $T \subset H \subset G$ and $\mathrm{Lie}(H) \subsetneq \mathfrak{g} = \mathrm{Lie}(G)$ (as $t \notin Z_G$). Thus $\dim H < \dim G$ and we can apply induction to claim that $T^H = H$. Again let $H^\times = H \setminus (H \cap Z_G)$.

One checks easily that $(T^\times)^H = H^\times$ and hence $(T^\times)^G = (H^\times)^G$.

Now let $\mathfrak{q} = (\mathrm{Lie}(H))^\perp \subset \mathrm{Lie}(G)$ $(\mathfrak{q} \neq 0)$

Alternately, $\mathrm{Lie}(H) = \mathrm{Ker}(\mathrm{Ad}(t) - 1)$ by definition and $\mathrm{Ad}(t)$ is orthogonal $\Rightarrow \mathfrak{q} = \mathrm{Im}(\mathrm{Ad}(t) - 1)$

Set $H_1 = \{ h \in H : \det((\mathrm{Ad}(h) - 1)|_{\mathfrak{q}}) \neq 0 \}$

- $H_1 \subset H$ is open
- $t \in H_1$
- $Z_G \cap H_1 = \phi$.

$\Rightarrow t \in H_1^G \subseteq (H^\times)^G = (T^\times)^G$

Thus it is enough to show that H_1^G is open in G.

Consider
$$\psi : \begin{array}{ccc} G \times H & \longrightarrow & G \\ (g, h) & \longmapsto & g h g^{-1} \end{array}$$

Easy check : $d\psi_{(g,h)}(X,Y) = \mathrm{Ad}(g) \cdot \left((\mathrm{Ad}(h^{-1}) - 1)(X) + Y \right)$

For $h \in H_1$, $\mathrm{Ad}(h^{-1}) - 1$ is invertible on $\mathfrak{q} = \mathrm{Lie}(H)^{\perp}$ and $Y \in \mathrm{Lie}(H)$ $\Rightarrow$ $d\psi$ is surjective at every point of $G \times H_1$ proving the claim. $\square$