# Lecture 27



(27.0) Let $G$ be a connected compact Lie group and $T \subset G$ a maximal torus. Last time we proved

(1) If $T' \subset G$ is another max'l torus, then $\exists\, g \in G$ s.t. $T' = gTg^{-1}$

(2) $\forall\, x \in G$, $\exists\, g \in G$ s.t. $gxg^{-1} \in T$.

Cor. (1) $\exp : \mathrm{Lie}(G) \longrightarrow G$ is surjective

(2) $Z_G \subset T$

(27.1) Prop. Let $S \subset G$ be a torus (i.e. connected, closed abelian subgroup of $G$) and $g \in G$ s.t. $gs = sg$ $\forall\, s \in S$. Then $\exists$ a torus $S'$ containing both $S$ and $g$.

Proof Let $A = \underline{\text{closure of the subgroup generated by } S \text{ \& } g}$

$$= \overline{\bigcup_{n \in \mathbb{Z}} g^n S}$$

$A_0 \subset A$ connected component containing $e$. Then $S \subset A_0$ and $\bigcup_{n \in \mathbb{Z}} g^n A_0 \subset A$ is open subgroup (hence closed). Thus $A = \bigcup_{n \in \mathbb{Z}} g^n A_0$.

But compactness of $A_0$ implies $\exists$ (smallest) $N > 0$ s.t. $g^N \in A_0$

$\Rightarrow$ $A/A_0$ is cyclic of order $N$.

Exercise: we can find $a \in A$ s.t $\{a^m,\ m \in \mathbb{Z}\} \subset A$ is dense.

By surjectivity of exp, $a = \exp(x)$ for some $x \in \mathrm{Lie}(G)$. ②

Hence $S' = \overline{\{\exp(tx) : t \in \mathbb{R}\}} \supset A$ and $S'$ is connected closed abelian subgroup (i.e. a torus) containing $S$ & $g$ $\square$

(27.2) Weyl group : define $W^{an} := N_G(T)/T$

$N_G(T)$ = normalizer of $T = \{g \in G : gTg^{-1} \subset T\}$

If $\mathfrak{t}_0 = \mathrm{Lie}(T)$, then $\mathrm{Ad}(g)\,\mathfrak{t}_0 \subset \mathfrak{t}_0 \quad \forall g \in N_G(T)$ and $\mathrm{Ad}(g)\big|_{\mathfrak{t}_0} = \mathrm{Id}_{\mathfrak{t}_0}$ for $g \in T$. That is,

$W^{an} \circlearrowright \mathfrak{t}_0$ (and hence on $\mathfrak{t}_0^*$, $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}_0$, ...)

faithfully

Lemma (a) $W^{an}$ is finite.

(b) Conjugacy classes in $G \longleftrightarrow T/W^{an}$

(c) $\{G$-invariant cnts. fns. on $G\} = \{W^{an}$-inv cnts. fns. on $T\}$

Proof. (a) We claim that $N_G(T)^\circ = T$.

$\uparrow$ connected component containing $e$

This will prove that $W^{an}$ is discrete. But $W^{an} \subset O(\mathfrak{h}_{\mathbb{R}})$ ← cpct

(w.r.t. positive definite form)

Hence $W^{an}$ is finite.

<u>$N_G(T)^\circ = T$</u> : Let $g \in N_G(T)^\circ$. By surjectivity of exp.

$g = \exp(x)$ for some $x \in \mathrm{Lie}(G)$. Then $\forall t \in \mathbb{R}$, $\exp(tx) \in N^\circ$

$\Rightarrow \quad ad(x)(\mathfrak{t}_0) \subset \mathfrak{t}_0 \quad \Rightarrow \quad x \in \mathfrak{t}_0 \quad \Rightarrow \quad g = \exp(x) \in T.$

~~(b) Obvious.~~ Below

(c) $f : G \to \mathbb{C}$ cnts and $f(gx g^{-1}) = f(x) \;\; \forall g \in G$ implies $f|_T : T \to \mathbb{C}$ (obviously cnts.) is $W^{an}$-invariant. This is clear since $\forall w \in W^{an}$ we can find a lift $\tilde{w} \in N_G(T)$ and $f(w \cdot t) = f(\tilde{w} t \tilde{w}^{-1}) = f(t)$.

Conversely, if $f : T \to \mathbb{C}$ (cnts & $W^{an}$-invariant) is given, define $F : G \to \mathbb{C}$ by: for $x \in G$, let $g \in G$ be s.t. $gxg^{-1} \in T$. Set

$$F(x) = f(gxg^{-1}) \qquad \text{(well-defined by (b))}.$$

It remains to check that $F$ is cnts. Let $g_n \to g$ in $G$. Write $g_n = x_n t_n x_n^{-1}$ $(x_n \in G,\ t_n \in T)$. By compactness, we can choose subsequences and assume $\begin{matrix} x_n \to x \in G \\ t_n \to t \in T \end{matrix}$ $\Rightarrow \; g = x t x^{-1}$. So

$$F(g_n) = F(t_n) = f(t_n) \longrightarrow f(t) = F(g).$$

(b) The only non-trivial part is to prove that if $s, t \in T$ are $G$-conj. $\exists\, g \in G$ s.t. $gtg^{-1} = s$, then they are $W$-conjugate. That is, if ~~then~~ we can find $g_0 \in N_G(T)$ s.t. $g_0 t g_0^{-1} = s$.

$Z_G(s) := \{ g' \in G : g's = sg' \} \supset Z_G(s)_0$ connected component of $e$

$\mathrm{Lie}(Z_G(s)_0) = \{ X \in \mathfrak{g}_0 : Ad(s)(X) = X \} \supset \mathfrak{t}_0$ and $Ad(g) \cdot \mathfrak{t}_0$ 2 max'l abelian subalgebras

$\Rightarrow \exists\, z \in Z_G(s_0)$ so that $\mathfrak{t}_0 = Ad(zg)\mathfrak{t}_0$. Hence $g_0 = zg \in N_G(T)$

and $zgtg^{-1}z^{-1} = zsz^{-1} = s$ $\square$

(27.3) Theorem. $W^{an} = W$ (Weyl group of the root system).

For this result the center plays no role, so we may assume $G$ is semisimple.

(Lecture 13 · page 6 · compact real forms).

$$\mathfrak{g}_0 = \mathrm{Lie}(G) \underset{\substack{\text{cpct} \\ \text{real form}}}{\subset} \mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

- $t_\alpha, h_\alpha \in \mathfrak{h}$ $\left( h_\alpha = \dfrac{2 t_\alpha}{|\alpha|^2} \right.$ · recall $|\alpha|^2 \in \mathbb{Q}$ $\left(\text{Lemma 13.6 page 5 of Lecture 13}\right)\Big)$

  $\mathfrak{h}_{\mathbb{R}}$ = real span of $h_\alpha$'s

  $\mathfrak{t}_0 = i\mathfrak{h}_{\mathbb{R}}$, we set $\sigma_\alpha^0 = i t_\alpha$

- $\{X_\alpha\}_{\alpha \in R}$ · Weyl basis. $\sigma_\alpha^+ = X_\alpha - X_{-\alpha}$ $\quad \sigma_\alpha^- = i(X_\alpha + X_{-\alpha}) \in \mathfrak{g}_0$

Commutation relations
$$[\sigma_\alpha^0, \sigma_\alpha^\pm] = \pm |\alpha|^2 \sigma_\alpha^\mp$$
$$[\sigma_\alpha^+, \sigma_\alpha^-] = 2\sigma_\alpha^0$$

$$\Rightarrow \quad r_\alpha = \exp\left(\frac{\pi}{\sqrt{2}\,|\alpha|} \cdot \sigma_\alpha^+\right) : \quad \begin{aligned} \sigma_\alpha^0 &\longmapsto -\sigma_\alpha^0 \\ x &\longmapsto x \quad \forall x \text{ s.t. } \alpha(x) = 0 \end{aligned}$$

So $r_\alpha \in N_G(T)$ and $r_\alpha = s_\alpha$ on $\mathfrak{h}_{\mathbb{R}}$. Hence $W \subset W^{an}$.

Conversely every element $g \in N_G(T)$ preserves the set of roots and hence the set of chambers $\mathcal{C} = \pi_0\left(\mathfrak{h}_{\mathbb{R}} \setminus \bigcup_{\alpha \in R} \mathrm{Ker}(\alpha)\right)$.

i.e. $w \in W^{an}$ acts as automorphism of $R$. But so does $W$ (Lecture 19 page 1)

$\Rightarrow \exists\ w_1 \in W$ s.t. $w w_1$ fixes the fundamental chamber $C_0$. ⑤

As an element of $W^{an}$ (since $W \subset W^{an}$ is already proved) we can lift it to an element $g \in N_G(T) \subset G$.

Pick a regular element, say $\delta = \frac{1}{2} \sum_{\substack{\alpha \in R_+ \\ (\text{w.r.t. } C_0)}} \alpha \in \mathfrak{h}_{\mathbb{R}}^*$ and corr. $h_\delta \in C_0$.

Then $g(h_\delta) = h_\delta$. But $Z_g(h_\delta) = \mathfrak{h}$, Hence $g \in T$ and we get $w_1 w = 1$. $\square$