

(28.0) Let  $G$  be a connected compact semisimple Lie group. Recall our notations:  $T \subset G$  is a max'l torus

$$\mathfrak{g}_0 = \text{Lie}(G) \quad \supset \quad \mathfrak{t}_0 = \text{Lie}(T)$$

$\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$  is a complex semisimple Lie algebra, so

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathbb{R}} \mathfrak{g}_{\alpha}$$

•  $\forall \alpha \in \mathbb{R}, t_{\alpha} \in \mathfrak{h}$  is uniquely determined by  $(t_{\alpha}, x) = \alpha(x) \quad \forall x \in \mathfrak{h}$

$$h_{\alpha} = \frac{2t_{\alpha}}{|\alpha|^2} \quad (\text{recall } |\alpha|^2 \in \mathbb{Q}).$$

$\mathfrak{h}_{\mathbb{R}}$  = real span of  $h_{\alpha}$ 's (or  $t_{\alpha}$ 's)

•  $\{X_{\alpha}\}_{\alpha \in \mathbb{R}}$ : Weyl basis.  $\sigma_{\alpha}^{+} = X_{\alpha} - X_{-\alpha}$   $\sigma_{\alpha}^{-} = i(X_{\alpha} + X_{-\alpha})$   
 $\sigma_{\alpha}^0 = it_{\alpha} \quad (\forall \alpha > 0)$

$$[\sigma_{\alpha}^0, \sigma_{\alpha}^{\pm}] = \pm |\alpha|^2 \sigma_{\alpha}^{\mp} \quad [\sigma_{\alpha}^{+}, \sigma_{\alpha}^{-}] = 2\sigma_{\alpha}^0$$

(eg.  $\mathfrak{g} = \mathfrak{sl}_2$ ,  $\{\sigma^0, \sigma^{\pm}\}$  is a basis of  $\mathfrak{su}(2) \subset \mathfrak{sl}_2$  with  $\sigma^0 = i\mathfrak{h}$   
 $\sigma^{+} = e - f$   
 $\sigma^{-} = i(e + f)$ )

$$[\sigma^0, \sigma^{\pm}] = \pm 2\sigma^{\mp} \quad [\sigma^{+}, \sigma^{-}] = 2\sigma^0$$

Note:  $\forall \alpha > 0$ ,  $\mathfrak{su}(2) \xrightarrow{i_{\alpha}} \mathfrak{g}_0$  is a Lie alg. hom.

$$\sigma^0 \longmapsto \frac{2}{|\alpha|^2} \sigma_{\alpha}^0$$

$$\sigma^{\pm} \longmapsto \frac{\sqrt{2}}{|\alpha|} \sigma_{\alpha}^{\pm}$$

(28.1) Definition:  $P := \{ \lambda \in \mathfrak{h}^* : \lambda(h_\alpha) = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \forall \alpha \in R \}$  (weight lattice) ②

(root lattice)  $Q := \sum_{\alpha \in R} \mathbb{Z}\alpha$ .  $P$  and  $Q$  are both free abelian groups in  $\mathfrak{h}_{\mathbb{R}}^*$  of full rank ( $= \dim \mathfrak{h} = \text{rank } \mathfrak{g}$ ), say  $l$ .

Clearly  $Q \subset P$ . More precisely, we have

Lemma. Let  $\{\alpha_1, \dots, \alpha_l\}$  be a set of simple roots. Define  $\omega_i \in \mathfrak{h}^*$  by  $\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$ . Then  $P = \sum_{j=1}^l \mathbb{Z}\omega_j$ . Moreover

$$Q = \sum_{i=1}^l \mathbb{Z}\alpha_i \quad \text{and} \quad \alpha_i = \sum_{j=1}^l c_{ij}\omega_j \Rightarrow c_{ij} = \alpha_i(h_j) = a_{ji} \in \mathbb{Z}$$

In particular,  $|P/Q| = \text{determinant of Cartan Matrix}$  ( $\det(A) > 0$  by Lecture 18 page 4)

pp.  $P = \sum_{j=1}^l \mathbb{Z}\omega_j$ :  $\forall \lambda \in P$ , let us write (since  $\{\omega_j\}$  are dual to a basis, they form a basis of  $\mathfrak{h}^*$ )

$$\lambda = \sum \lambda_i \omega_i. \quad \text{Then } \lambda_i = \lambda(h_i) \in \mathbb{Z} \Rightarrow P \subset \sum_{i=1}^l \mathbb{Z}\omega_i.$$

Conversely if  $\lambda \in \sum_{i=1}^l \mathbb{Z}\omega_i$  (that is,  $\lambda_i = \lambda(h_i) \in \mathbb{Z} \forall i$ )

We prove by induction on  $\text{ht}(\alpha)$ , that  $\forall \alpha > 0$ ,  $\lambda(h_\alpha) \in \mathbb{Z}$  (recall that for  $\alpha = \sum n_i \alpha_i$ ,  $\text{ht}(\alpha) := \sum n_i \alpha_i$ ).  $\text{ht}(\alpha) = 1 \Rightarrow \alpha = \alpha_j$  &  $\lambda(h_j) \in \mathbb{Z}$  is given.

For  $\alpha > 0$ ,  $\text{ht}(\alpha) > 1$ , let  $i \in \{1, \dots, l\}$  be st.  $s_i(\alpha) > 0$  and  $\text{ht}(s_i \alpha) < \text{ht}(\alpha)$

$$\text{Then } \beta = s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$$

$$\Rightarrow \frac{2(\lambda, \beta)}{(\beta, \beta)} = \frac{2(\lambda, \alpha)}{|\beta|^2} - \frac{\alpha(h_i) \cdot 2(\lambda, \alpha_i)}{|\beta|^2}$$

As  $\beta = S_i \alpha$ ,  $|\beta|^2 = |\alpha|^2$  and we get

(3)

$$\frac{2(\lambda, \alpha)}{|\alpha|^2} = \frac{2(\lambda, \beta)}{|\beta|^2} - \frac{2(\alpha, \alpha_i)}{|\alpha|^2} \frac{2(\lambda, \alpha_i)}{|\alpha_i|^2} \in \mathbb{Z}.$$

(28.2) Definition (analytic weight lattice)  $\mathcal{P}^{an}$ . Let  $\lambda \in \mathfrak{h}^*$ . We say  $\lambda$  is analytic weight if  $\lambda|_{\mathfrak{t}_0} : \mathfrak{t}_0 \rightarrow \mathbb{C}$  is differential of a character  $\xi_\lambda : T \rightarrow S^1$  (i.e. a continuous, hence smooth - see Prop. 7.1 of Lecture 7, page 2, group hom.).

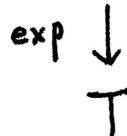
In this case  $\lambda|_{\mathfrak{t}_0} : \mathfrak{t}_0 \rightarrow i\mathbb{R}$  and hence  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  (real).

Prop. (1)  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  extended to  $\mathfrak{h}^*$  (over  $\mathbb{C}$ ) is analytic weight iff  $\lambda(x) \in 2\pi i \mathbb{Z}$ . (\*)

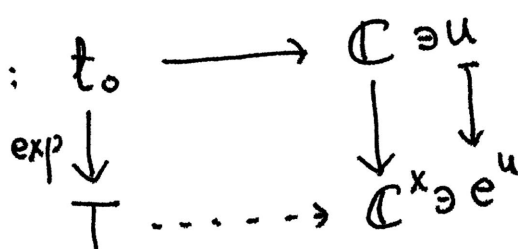
(2)  $Q \subset \mathcal{P}^{an} \subset \mathcal{P}$

Pf. (1)  $(\Rightarrow)$  clear since  $\xi_\lambda(\exp(x)) = e^{\lambda(x)}$ .

$(\Leftarrow)$  We have  $\mathfrak{t}_0 (\cong \mathbb{R}^n)$ . Since  $T$  is abelian,  $\exp$  is a



group hom. realizing  $\mathfrak{t}_0$  as the universal cover of  $T$ . Now a group homomorphism  $\lambda : \mathfrak{t}_0 \rightarrow \mathbb{C} \ni u$  descends



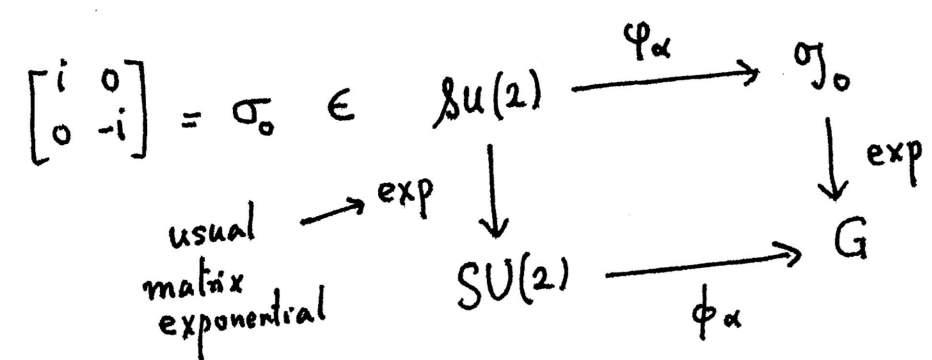
to a gp. hom  $\xi_\lambda : T \rightarrow \mathbb{C}^\times$  iff (\*) holds. In this case  $\text{Im}(\xi_\lambda) = S^1$  and hence  $\text{Image}(\lambda) \subset i\mathbb{R}$ .

(only cpct subgp. of  $\mathbb{C}^\times$ )

(2)  $Q \subset P^{an}$ . It is enough to show that each  $\alpha \in R$  is analytically (integrat) weight. But  $T \subset G \hookrightarrow \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$   
 so the action of  $T$  on 1-dim'l  $\mathbb{C}$ -vector space  $\mathfrak{g}_\alpha$  is through a character  $\xi_\alpha: T \rightarrow S^1$  whose differential is  $\alpha$ .

$P^{an} \subset P$ . Let  $\lambda: \mathfrak{h} \rightarrow \mathbb{C}$  ( $\lambda(t_0) \in i\mathbb{R}$ ) be analytically weight.

Let  $\alpha \in R$  and consider  $\mathfrak{su}(2) \xrightarrow{\varphi_\alpha} \mathfrak{g}_\alpha$  ((28.0) above)  
 Since  $SU(2)$  is simply-connected, we get  $\phi_\alpha: SU(2) \rightarrow G$  whose differential is  $\varphi_\alpha$ .

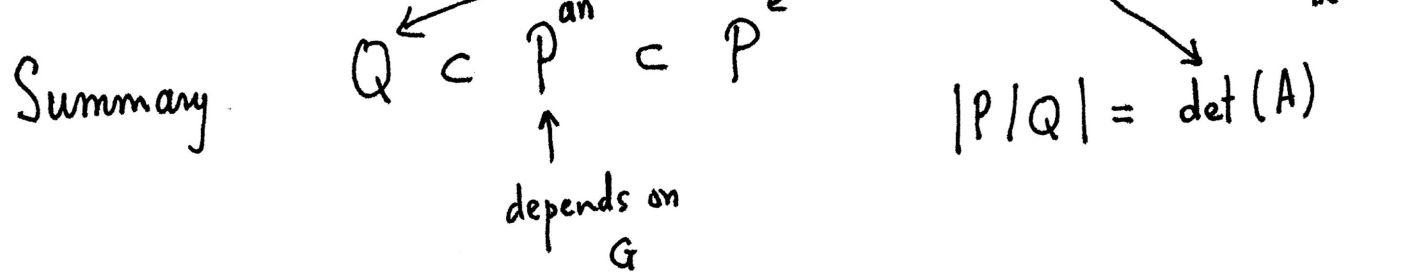


$\Rightarrow \exp(2\pi\sigma_0) = 1 \Rightarrow \exp(\varphi_\alpha(2\pi\sigma_0)) = 1$

$\varphi_\alpha(2\pi\sigma_0) = 2\pi \cdot \frac{2}{|\alpha|^2} i t_\alpha = 2\pi i h_\alpha$

So  $\exp(2\pi i h_\alpha) = 1$ . Since  $\lambda$  is analytic weight we get by defn,  $\lambda(2\pi i h_\alpha) \in 2\pi i \mathbb{Z}$  i.e.  $\lambda(h_\alpha) \in \mathbb{Z} \quad \forall \alpha \in R$

Hence  $\lambda \in P$ .



(28.3) Prop. Let  $\tilde{G} \xrightarrow{1} G$  be a finite covering (hence  $\tilde{G}$  is also compact and  $\text{Lie}(\tilde{G}) = \text{Lie}(G)$  is also semisimple). Then  $\text{Ker } \eta$  equals  $\mathcal{P}^{\text{an}}(\tilde{G}) / \mathcal{P}^{\text{an}}(G)$ . (5)

Pf.  $\text{Ker}(\eta)$  is discrete normal, hence a subgroup of  $Z_{\tilde{G}} \subset \tilde{T}$  (max'l torus)

We take  $T \subset G$  max'l torus = Image of  $\tilde{T}$  under  $\eta$ :

$$\eta: \tilde{T} \rightarrow T \quad \text{Now } \mathcal{P}^{\text{an}}(G) \leftrightarrow \text{Characters of } T \ni \sum$$

$$\mathcal{P}^{\text{an}}(\tilde{G}) \leftrightarrow \text{Characters of } \tilde{T} \ni \sum \circ \eta$$

and the prop. follows □

(28.4) Thm (Weyl) Let  $G$  be a connected compact semisimple Lie group. Then  $\pi_1(G)$  is finite.

Lemma:  $G$ : connected compact Lie gp. Then  $\pi_1(G)$  is f.g. abelian.

Proof of Thm. Let  $\tilde{G}$  be the universal covering group, so that:

$\pi_1(G) = \text{Ker}(\eta) \subset Z_{\tilde{G}}$ .  $\text{Ker}(\eta)$  is then f.g. by Lemma. If it is not finite we can find  $Z_1 \subset \text{Ker}(\eta)$  of finite index  $> \det(A)$ .

Then  $\tilde{G}/Z_1$  is finite cover w/ number of sheets  $> \det(A)$

$$\downarrow$$

$$G \quad \parallel \quad |\mathcal{P}^{\text{an}}(\tilde{G}/Z_1) / \mathcal{P}^{\text{an}}(G)|$$

But  $Q \subset \mathcal{P}^{\text{an}}(G) \subset \mathcal{P}^{\text{an}}(\tilde{G}/Z_1) \subset P$  and  $|P/Q| = \det(A)$  contradiction □

(28.5) Proof of Lemma.  $G =$  connected cpct. Lie group. ⑥

Claim:  $\exists U \subset \tilde{G}$  open st.  
 (i)  $\tilde{e} \in U$  (ii)  $U = \bar{U}^{-1}$  (iii)  $\bar{U}$  is cpct  
 (iv)  $\eta(U) = G$

(Universal cover)

$$\begin{array}{c} \tilde{G} \supset Z = \text{Ker}(\eta) \\ \eta \downarrow \\ G \end{array}$$

Assuming the claim, (iv)  $\Rightarrow \tilde{G} = ZU$ ,  $\bar{U}\bar{U}^{-1}$  is cpct (by (iii)) hence

$$(**) - \bar{U}\bar{U}^{-1} = \bigcup_{j=1}^k z_j U \quad (z_1 \dots z_k \in Z).$$

Let  $Z_1 =$  subgp. of  $Z$  generated by  $z_1 \dots z_k$  and consider  $\tilde{G} \xrightarrow{\text{pr}} \tilde{G}/Z_1$

Let  $E = \text{pr}(\bar{U})$ . Then  $E$  is a cpct subgp. of  $\tilde{G}/Z_1$  (from (i) (ii) and

(\*\*)) and  $E$  is then a finite cover (note  $E \supset$  image of  $U$ )

$$\downarrow \\ G$$

and is a subgp. so by connectedness  $E = \tilde{G}/Z_1$ . Hence  $G$  has a

finite cover whose fundamental gp. is f.g.  $\Rightarrow \pi_1(G)$  is f.g.

Proof of the claim:  $\forall x \in G$ , we can find an open nhd  $V_x$  evenly covered by  $\eta$ , and  $x \in V'_x \subset V_x$  s.t.  $\bar{V}'_x \subset V_x$ . (may assume these open sets are connected & simply connected). By cpctness of  $G$ , we can cover  $G$

with finitely many  $V'_{x_1}, \dots, V'_{x_p}$ . Choose a connected component

$W_{x_j} (\supset W'_{x_j})$  in  $\eta^{-1}(V_{x_j}) (\supset \eta^{-1}(V'_{x_j}))$ . (for the open set containing  $e$ , pick the sheet containing  $\tilde{e}$ ). Let  $U = \bigcup_{j=1}^p W_{x_j}$ . We may have to enlarge

it to make sure  $U = \bar{U}^{-1}$ . All the properties listed in the claim follow

trivially. □