Lecture 28

(28.0) Let G be a connected compact semisimple Lie group. Recall our notations : $T \subset G$ is a max'l torus

$$\mathfrak{g}_0 = \mathrm{Lie}(G) \supset \mathfrak{t}_0 = \mathrm{Lie}(T)$$

$\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a complex semisimple Lie algebra, so

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

- $\forall \alpha \in R$, $t_\alpha \in \mathfrak{h}$ is uniquely determined by $(t_\alpha, x) = \alpha(x)\ \forall x \in \mathfrak{h}$

$$h_\alpha = \frac{2t_\alpha}{|\alpha|^2} \quad (\text{recall } |\alpha|^2 \in \mathbb{Q}).$$

$\mathfrak{h}_{\mathbb{R}}$ = real span of h_α's (or t_α's)

- $\{X_\alpha\}_{\alpha \in R}$: Weyl basis. $\quad \sigma_\alpha^+ = X_\alpha - X_{-\alpha} \quad \sigma_\alpha^- = i(X_\alpha + X_{-\alpha})$

$\sigma_\alpha^0 = it_\alpha \quad (\forall \alpha > 0)$

$$[\sigma_\alpha^0, \sigma_\alpha^\pm] = \pm |\alpha|^2 \sigma_\alpha^\mp \qquad [\sigma_\alpha^+, \sigma_\alpha^-] = 2\sigma_\alpha^0$$

(e.g. $\mathfrak{g} = \mathfrak{sl}_2$, $\{\sigma^0, \sigma^\pm\}$ is a basis of $\mathfrak{su}(2) \subset \mathfrak{sl}_2$ with $\sigma^0 = ih$, $\sigma^+ = e - f$, $\sigma^- = i(e+f)$

$$[\sigma^0, \sigma^\pm] = \pm 2\sigma^\mp \qquad [\sigma^+, \sigma^-] = 2\sigma^0 \text{)}$$

Note : $\forall \alpha > 0$, $\quad \mathfrak{su}(2) \xrightarrow{i_\alpha} \mathfrak{g}_0$ is a Lie alg. hom.

$$\sigma^0 \longmapsto \frac{2}{|\alpha|^2}\sigma_\alpha^0$$

$$\sigma^\pm \longmapsto \frac{\sqrt{2}}{|\alpha|}\sigma_\alpha^\pm$$

(28.1) Definition: $P := \{ \lambda \in \mathfrak{h}^* : \lambda(h_\alpha) = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \ \forall \alpha \in R \}$ (weight lattice) ②

(root lattice) $Q := \sum_{\alpha \in R} \mathbb{Z}\alpha$. P and Q are both free abelian groups in $\mathfrak{h}^*_{\mathbb{R}}$ of full rank $(= \dim \mathfrak{h} = \operatorname{rank} \mathfrak{g})$, say ℓ.

Clearly $Q \subset P$. More precisely, we have

Lemma. Let $\{\alpha_1, \dots, \alpha_\ell\}$ be a set of simple roots. Define $\omega_i \in \mathfrak{h}^*$ by $\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$. Then $P = \sum_{j=1}^{\ell} \mathbb{Z}\omega_j$. Moreover

$Q = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i$ and $\alpha_i = \sum_{j=1}^{\ell} c_{ij}\omega_j \Rightarrow c_{ij} = \alpha_i(h_j) = a_{ji} \in \mathbb{Z}$.

In particular, $|P/Q|$ = determinant of Cartan Matrix (det(A)>0 by Lecture 18 page 4)

Pf. $\underline{P = \sum_{j=1}^{\ell} \mathbb{Z}\omega_j}$: $\forall \lambda \in P$, let us write (since $\{\omega_j\}$ are dual to a basis, they form a basis of $\mathfrak{h}^*$)

$\lambda = \sum \lambda_i \omega_i$. Then $\lambda_i = \lambda(h_i) \in \mathbb{Z} \Rightarrow P \subset \sum_{i=1}^{\ell} \mathbb{Z}\omega_i$.

Conversely if $\lambda \in \sum_{i=1}^{\ell} \mathbb{Z}\omega_i$ (that is, $\lambda_i = \lambda(h_i) \in \mathbb{Z} \ \forall i$)

We prove by induction on $ht(\alpha)$, that $\forall \alpha > 0$, $\lambda(h_\alpha) \in \mathbb{Z}$ (recall that for $\alpha = \sum n_i \alpha_i$, $ht(\alpha) := \sum n_i \alpha_i$). $ht(\alpha) = 1 \Rightarrow \alpha = \alpha_j$ & $\lambda(h_j) \in \mathbb{Z}$ is given.

For $\alpha > 0$, $ht(\alpha) > 1$, let $i \in \{1, \dots, \ell\}$ be s.t. $s_i(\alpha) > 0$ and $ht(s_i\alpha) < ht(\alpha)$

Then $\beta = s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$

$$\Rightarrow \frac{2(\lambda,\beta)}{(\beta,\beta)} = \frac{2(\lambda,\alpha)}{|\beta|^2} - \alpha(h_i)\frac{2(\lambda,\alpha_i)}{|\beta|^2}$$

As $\beta = s_i \alpha$, $|\beta|^2 = |\alpha|^2$ and we get

$$\frac{2(\lambda,\alpha)}{|\alpha|^2} = \frac{2(\lambda,\beta)}{|\beta|^2} - \frac{2(\alpha,\alpha_i)}{|\alpha|^2}\,\frac{2(\lambda,\alpha_i)}{|\alpha_i|^2} \in \mathbb{Z}. \qquad \square$$

(28.2) Definition (analytic weight lattice) P^{an}. Let $\lambda \in \mathfrak{h}^*$. We say λ is <u>analytic weight</u> if $\lambda|_{\mathfrak{t}_0} : \mathfrak{t}_0 \to \mathbb{C}$ is differential of a character $\xi_\lambda : T \to S^1$ (i.e. a continuous, hence smooth - see Prop. 7.1 of Lecture 7, page 2, group hom.).

In this case $\lambda|_{\mathfrak{t}_0} : \mathfrak{t}_0 \longrightarrow i\mathbb{R}$ and hence $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ (real).

Prop. (1) $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ extended to $\mathfrak{h}^*$ (over $\mathbb{C}$) is analytic weight iff

$$X \in \mathfrak{t}_0 \text{ s.t. } \exp(X) = 1 \implies \lambda(X) \in 2\pi i \mathbb{Z}. \qquad (*)$$

(2) $Q \subset P^{an} \subset P$

Pf. (1) ($\Rightarrow$) clear since $\xi_\lambda(\exp(X)) = e^{\lambda(X)}$.

($\Leftarrow$) We have $\mathfrak{t}_0 \ (\simeq \mathbb{R}^n)$. Since T is abelian, exp is a

$$\begin{array}{c} \mathfrak{t}_0 \\ \big\downarrow{\scriptstyle \exp} \\ T \end{array}$$

group hom. realizing $\mathfrak{t}_0$ as the universal cover of T. Now a group homomorphism

$$\begin{tikzcd} \lambda : \mathfrak{t}_0 \arrow[r] \arrow[d, "\exp"'] & \mathbb{C} \ni u \arrow[d] \\ T \arrow[r, dashed] & \mathbb{C}^\times \ni e^u \end{tikzcd}$$

descends to a gp. hom $\xi_\lambda : T \to \mathbb{C}^\times$ iff $(*)$ holds. In this case $\mathrm{Im}(\xi_\lambda) = S^1$ and hence $\mathrm{Image}(\lambda) \subset i\mathbb{R}$.

(only cpct subgp. of $\mathbb{C}^\times$)

(2) $\underline{Q \subset P^{an}}$. It is enough to show that each $\alpha \in R$ is analytic~~ally integral~~ weight. But $T \subset G \circlearrowright \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

so the action of T on 1-dim'l $\mathbb{C}$-vector space $\mathfrak{g}_\alpha$ is through a character $\xi_\alpha : T \to S^1$ whose differential is α.

$\underline{P^{an} \subset P}$. Let $\lambda : \mathfrak{h} \to \mathbb{C}$ $\left(\lambda(\mathfrak{t}_0) \subset i\mathbb{R} \right)$ be analytic~~ally~~ weight.

Let $\alpha \in R$ and consider $\mathfrak{su}(2) \xrightarrow{\varphi_\alpha} \mathfrak{g}_0$ ((28.0) above)

Since $SU(2)$ is simply-connected, we get $\phi_\alpha : SU(2) \to G$ whose differential is φ_α.

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \sigma_0 \in \begin{tikzcd} \mathfrak{su}(2) \arrow[r, "\varphi_\alpha"] \arrow[d, "\exp"] & \mathfrak{g}_0 \arrow[d, "\exp"] \\ SU(2) \arrow[r, "\phi_\alpha"'] & G \end{tikzcd}$$

usual matrix exponential $\to$ exp

$$\Rightarrow \quad \exp(2\pi\sigma_0) = 1 \quad \Rightarrow \quad \exp\left(\varphi_\alpha(2\pi\sigma_0)\right) = 1$$

$$\varphi_\alpha(2\pi\sigma^0) = 2\pi \cdot \frac{2}{|\alpha|^2} i t_\alpha = 2\pi i h_\alpha$$

So $\exp(2\pi i h_\alpha) = 1$. Since λ is analytic weight we get by defn, $\lambda(2\pi i h_\alpha) \in 2\pi i \mathbb{Z}$ i.e. $\lambda(h_\alpha) \in \mathbb{Z}$ $\forall \alpha \in R$

Hence $\lambda \in P$. $\square$

depend only on $\mathfrak{g} \cong \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$

Summary $\quad Q \subset P^{an} \subset P \qquad |P/Q| = \det(A)$

$\uparrow$ depends on G

(28.3) Prop. Let $\tilde{G} \xrightarrow{\eta} G$ be a finite covering (hence $\tilde{G}$ is also compact and $\mathrm{Lie}(\tilde{G}) = \mathrm{Lie}(G)$ is also semisimple). Then $\mathrm{Ker}\,\eta$ equals

$$P^{an}(\tilde{G}) / P^{an}(G).$$

Pf. $\mathrm{Ker}(\eta)$ is discrete normal, hence a subgp. of $Z_{\tilde{G}} \subset \tilde{T}$ (max'l torus)

We take $T \subset G$ max'l torus = Image of $\tilde{T}$ under η:

$\eta : \tilde{T} \to T$. Now

$$\begin{array}{ccccc} P^{an}(G) & \longleftrightarrow & \text{Characters of } T & \ni & \xi \\ \cap & & \cap & & \downarrow \\ P^{an}(\tilde{G}) & \longleftrightarrow & \text{Characters of } \tilde{T} & \ni & \xi \circ \eta \end{array}$$

and the prop. follows $\square$

(28.4) <u>Thm</u> (Weyl) Let G be a connected compact semisimple Lie group. Then $\pi_1(G)$ is finite.

<u>Lemma</u>: G: connected compact Lie gp. Then $\pi_1(G)$ is f.g. abelian.

<u>Proof of Thm.</u> Let $\begin{array}{c}\tilde{G}\\ \eta\downarrow \\ G\end{array}$ be the universal covering group, so that

$\pi_1(G) = \mathrm{Ker}(\eta) \subset Z_{\tilde{G}}$. $\mathrm{Ker}(\eta)$ is then f.g. by Lemma. If it is not finite we can find $Z_1 \subset \mathrm{Ker}(\eta)$ of finite index $> \det(A)$.

Then $\begin{array}{c}\tilde{G}/Z_1\\ \downarrow \\ G\end{array}$ is finite cover w/ number of sheets $> \det(A)$

$$\text{number of sheets} = |P^{an}(\tilde{G}/Z_1) / P^{an}(G)|$$

But $Q \subset P^{an}(G) \subset P^{an}(\tilde{G}/Z_1) \subset P$ and $|P/Q| = \det(A)$ contradiction $\square$

(28.5) Proof of Lemma. G = connected cpct. Lie group.

Claim: $\exists\, U \subset \tilde{G}$ open s.t.

(i) $\tilde{e} \in U$ (ii) $U = U^{-1}$ (iii) $\overline{U}$ is cpct

(iv) $\eta(U) = G$

(Universal cover)
$$\begin{array}{c} \tilde{G} \supset Z = \mathrm{Ker}(\eta). \\ \eta \downarrow \quad \\ G \quad \end{array}$$

Assuming the claim, (iv) $\Rightarrow$ $\tilde{G} = ZU$, $\overline{U}\,\overline{U}^{-1}$ is cpct (by (iii)) hence

$$(**) \quad - \quad \overline{U}\,\overline{U}^{-1} \subset \bigcup_{j=1}^{K} z_j U \qquad (z_1 \ldots z_K \in Z).$$

Let Z_1 = subgp. of Z generated by $z_1 \ldots z_K$ and consider $\tilde{G} \xrightarrow{pr} \tilde{G}/Z_1$

Let $E = pr(\overline{U})$. Then E is a cpct subgp. of $\tilde{G}/Z_1$ (from (i) (ii) and (**)) and $\begin{array}{c} E \\ \downarrow \\ G \end{array}$ is then a finite cover (note $E \supset$ image of U and is a subgp. so by connectedness $E = \tilde{G}/Z_1$). Hence G has a finite cover whose fundamental gp. is f.g. $\Rightarrow \pi_1(G)$ is f.g.

Proof of the claim: $\forall\, x \in G$, we can find an open nbd V_x evenly covered by η, and $x \in V_x' \subsetneq V_x$ s.t. $\overline{V_x'} \subset V_x$. (may assume these open sets are connected & simply connected). By cpctness of G, we can cover G with finitely many $V_{x_1}', \ldots, V_{x_p}'$. Choose a connected component W_{x_j} ($\supset W_{x_j}'$) in $\eta^{-1}(V_{x_j})$ ($\supset \eta^{-1}(V_{x_j}')$). (for the open set containing e, pick the sheet containing $\tilde{e}$). Let $U = \bigcup_{j=1}^{p} W_{x_j}'$. We may have to enlarge it to make sure $U = U^{-1}$. All the properties listed in the claim follow trivially. □