Lecture 29

(29.0) Summary of results so far :

- Lie's Theorems : we have a functor Lie : Lie Gps $\longrightarrow$ Lie Alg.

  Moreover $\quad \underset{\text{Lie Gp}}{\mathrm{Hom}}(G, G') = \underset{\text{Lie Alg}}{\mathrm{Hom}}(\mathfrak{g}, \mathfrak{g}')$ if G is simply-connected.

- Fundamental Group. G = a connected Lie group, then

$$\pi_1(G) = \mathrm{Ker}(\underbrace{\tilde{G}}_{\text{universal cover}} \longrightarrow G) \subset \text{Center of } \tilde{G}.$$

- $\mathbb{C}$ - semisimple Lie alg. $\longleftrightarrow$ Root Systems $\longleftrightarrow$ Dynkin diag.

  $\updownarrow$

  compact real forms

- G = cpct. connected Lie gp. $\Longrightarrow$ $\underset{\mathfrak{g}_0}{\overset{\|}{\mathrm{Lie}(G)}} = \underset{\text{abelian}}{\underset{\uparrow}{\mathfrak{z}}} \oplus \underset{\substack{\text{semisimple} \\ \text{(cpct real form)}}}{\underset{\uparrow}{[\mathfrak{g}_0, \mathfrak{g}_0]}}$

$$G = \underset{\text{a torus}}{\underset{\uparrow}{Z^0}} \cdot \underset{\substack{\text{cpct connected} \\ \text{s.s. Lie gp.}}}{\underset{\uparrow}{G_{ss}}} \quad \text{s.t.} \quad \underset{\substack{\text{cpct real} \\ \text{form} \\ \text{(unique)}}}{\underset{\uparrow}{\mathrm{Lie}(G_{ss})}} < \underset{\mathbb{C}\text{- s.s. Lie alg}}{\underset{\uparrow}{\mathrm{Lie}(G_{ss}) \otimes_{\mathbb{R}} \mathbb{C}}}$$

- Conversely X = a Dynkin diagram

$$\rightsquigarrow \quad \mathfrak{g} = \text{corr } \mathbb{C}\text{- s.s. Lie algebra}$$
$$\cup$$
$$\mathfrak{g}_0 \quad \text{unique cpct real form}$$

$G_0 = \mathrm{Aut}(\mathfrak{g}_0)^\circ$ is cpct and hence $\pi_1(G_0)$ is finite so if $\begin{array}{c}\widetilde{G}\\ \downarrow\\ G_0\end{array}$ is universal cover, $\widetilde{G}$ is also cpct.

Hence if G is any Lie group (connected) with Lie alg. $\mathfrak{g}_0$, we have a covering map $\widetilde{G} \longrightarrow G \Rightarrow G$ is cpct.

Extreme case: $G = \widetilde{G}/Z_{\widetilde{G}}$ so $|Z_{\widetilde{G}}| < \infty$ (in fact $|Z_{\widetilde{G}}| \leq \det(A)$.

$Q \subset P^{an} \subset P$: for $\widetilde{G}$, $P^{an} = P$ and $Z_{\widetilde{G}} (\simeq P/Q)$, $\pi_1 = \{e\}$

for $\widetilde{G}/Z_{\widetilde{G}} = G$, $Q = P^{an}$ and $Z_G = \{e\}$, $\pi_1 (\simeq P/Q)$.

We used heavily

- Peter-Weyl Theorem. μ = the unique left (& right) inv. measure on Borel σ-alg. of G s.t. $\mu(G) = 1$

$$L^2(G) \underset{\uparrow}{=} \widehat{\bigoplus_{\lambda \in \Lambda_G}} V_\lambda^{\oplus d_\lambda}$$

(as G-representations)

: Λ_G = set of iso. classes of f.d. irred. repns. of G

$d_\lambda = \dim V_\lambda$

(The map is matrix-coefficients:

$$V_\lambda^{\oplus d_\lambda} \simeq V_\lambda \otimes V_\lambda \longrightarrow L^2(G)$$

$$(v, w) \longmapsto \Phi^{V_\lambda}_{v,w}(g) = (g \cdot v, w) \Big)$$

(29.1) More questions:

- What is the set Λ_G ?
- For $\lambda \in \Lambda_G$, how do we construct V_λ ? More specifically do we know its associated character, i.e.

$$\chi_{V_\lambda} : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \mathrm{Tr}_{V_\lambda}(g \text{ acting on } V_\lambda) \ ?$$

χ_{V_λ} is G-invariant smooth function on G, hence can be viewed as W-inv. smooth fn. on $T \subset G$.

As $T \circlearrowright V_\lambda$ via mutually commuting Hermitian matrices V_λ decomposes into "weight spaces"

$$\forall\ \xi : T \to S^1, \quad V_\lambda[\xi] = \left\{ v \in V_\lambda : \begin{array}{l} t.v = \xi(t) v \\ \forall\, t \in T \end{array} \right\}$$

Then
$$\chi_{V_\lambda}(t) = \sum_{\substack{\xi : T \to S^1 \\ \text{characters}}} \dim(V_\lambda[\xi]) . \xi(t) \qquad \text{---} (*)$$

characters ⟵ also denoted by P^{an}.

Thus asking for χ_{V_λ} is more general than asking for $\dim V_\lambda$. (but not as hard as asking for explicit construction of V_λ).

(29.2) Usually we write the eigenvalues of $T \circlearrowright V$ additively.

$$\left\{ \begin{matrix} \text{Characters} \\ T \to S^1 \end{matrix} \right\} \longleftrightarrow P^{an} \subset \mathfrak{t}_0^*$$

$$\begin{matrix} \Psi & & \Psi \text{ lattice} \\ \xi_\mu & \longleftrightarrow & \mu \end{matrix}$$

s.t. μ is diff'l of ξ_μ, i.e. $\left(e^{\mu(x)} = \xi_\mu(\exp(x))\right)$

We write $V_\lambda[\mu]$ for $V_\lambda[\xi_\mu]$ in (*).

(29.3) Answers. [Weyl]

- $\Lambda_G = P_+^{an} = \{\lambda \in P^{an} : \lambda(h_\alpha) \geq 0 \ \forall \alpha \in R_+\}$
  dominant weights

- $\forall \lambda \in P_+^{an}$

$$\chi_\lambda(t) = \frac{\sum_{w \in W} \varepsilon(w)\, \xi_{w \cdot \lambda}(t)}{\prod_{\alpha \in R_+} (1 - \xi_{-\alpha}(t))}$$

[Weyl Character formula]

Here let $\delta \in \mathfrak{h}_{\mathbb{R}}^*$ be given by $\delta = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \ \left(= \sum_{j=1}^{\ell} \omega_j\right)$

Shifted action $w \cdot \lambda = w(\lambda + \delta) - \delta$

(29.4) Example of $\mathfrak{sl}_2(\mathbb{C})$. We have seen already

Irred.- f.d. repns. of $\mathfrak{sl}_2\mathbb{C} \longleftrightarrow \lambda \in \mathbb{N}$

V_λ ($\lambda \in \mathbb{N}$) has basis $\{v_0, v_1, \dots, v_\lambda\}$

$$h.v_j = (\lambda - 2j)\,v_j \qquad f v_j = v_{j+1}$$

$$e v_j = (\lambda - j + 1)\, j \, v_{j-1}.$$

$SU(2) \circlearrowright V_\lambda$ $\quad T = S^1$ acts by

$$t.v_j = \xi_\lambda(t) \cdot \sum_{j=0}^{\lambda} e^{-2jt} = e^{\lambda t}\left(\frac{1 - e^{-2(\lambda+1)t}}{1 - e^{-2t}}\right)$$

$$= \frac{e^{\lambda t} - e^{(-(\lambda+1)-1)t}}{1 - e^{-2t}}$$

$$= \frac{\xi_\lambda(t) - \xi_{-\lambda-2}(t)}{1 - \xi_{-2}(t)}$$

(29.5) Both of these assertions follow from Peter-Weyl and the following Weyl's integration formula: (for integrable f)

$$|W| \cdot \int_G f(x)\, d\mu^G(x) = \int_T \left[\int_{G/T} f(g t g^{-1})\, d\mu^{G/T}(\bar{g}) \right] |D(t)|^2 \, d\mu^T(t)$$

- $|D(t)|^2 = \prod_{\alpha \in R} |1 - \xi_\alpha(t^{-1})|^2$
- μ^G, μ^T - Haar measures ($\mu^G(G) = \mu^T(T) = 1$)
- $\mu^{G/T}$ G-inv. measure

are chosen so that

$$\int_G f(x)\, d\mu^G(x) = \int_{G/T} \left(\int f(xt)\, d\mu^T(t) \right) d\mu^{G/T}(\bar{x})$$