Lecture 30

(30.0) G : connected compact Lie group

$T \subset G$ maximal torus. $\mathfrak{g}_0 = \mathrm{Lie}\, G \supset \mathfrak{t}_0 = \mathrm{Lie}(T)$

$W = N_G(T)/T$ Weyl group.

Recall $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{z} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

$\left(\{\alpha_1, \dots, \alpha_\ell\} \subset R \text{ base} \right)$

Definition: $\delta := \frac{1}{2} \sum_{\alpha \in R_+} \alpha$

$$\Delta(x) := \prod_{\alpha \in R_+} \left(e^{\alpha(x)/2} - e^{-\alpha(x)/2} \right) \quad \forall\, x \in \mathfrak{t}_0$$

(30.1) Lemma: (1) $\delta(h_i) = 1 \;\; \forall\, i = 1, \dots, \ell$. Thus $\delta = \sum_{i=1}^{\ell} \omega_i$ where $\{\omega_i\}$ are the fundamental weights ($\omega_i(h_j) = \delta_{ij}$).

(2) Δ descends to a function on T iff $x \mapsto e^{\delta(x)}$ does. $\left(\Delta = e^{\delta} \cdot \prod_{\alpha \in R_+} (1 - e^{-\alpha}) \right)$.

(3) $|\Delta(x)|^2 = \prod_{\alpha \in R_+} |1 - e^{-\alpha(x)}|^2$ descends to T.

Proof. (1) We use the fact that $\forall\, i = 1, \dots, \ell$; $\alpha \in R_+$

$s_i(\alpha) < 0 \iff \alpha = \alpha_i$ (Prop. 19.4 page 3)

So $s_i(\delta) = \frac{1}{2} \sum_{\substack{\alpha \in R_+ \\ \alpha \neq \alpha_i}} s_i(\alpha) - \frac{1}{2}\alpha_i = \delta - \alpha_i$

By definition $s_i(\delta) = \delta - \delta(h_i)\alpha_i = \delta - \alpha_i$

$\Rightarrow \delta(h_i) = 1.$

(2) Recall that $x \mapsto e^{\alpha(x)}$ $(x \in \mathfrak{t}_0)$ descends to a function $\xi_\alpha : T \to S^1$ (Prop. 28.2 page 3). The result then follows from

$$\Delta(x) = \prod_{\alpha \in R_+} e^{\frac{\alpha(x)}{2}} \cdot \prod_{\alpha \in R_+} 1 - e^{-\alpha(x)}$$

(3) $$|\Delta(x)|^2 = \prod_{\alpha \in R_+} |e^{\alpha(x)/2}|^2 \cdot \prod_{\alpha \in R_+} |1 - e^{-\alpha(x)}|^2$$

$(x \in \mathfrak{t}_0$, hence $\alpha(x) \in i\mathbb{R}$ $\forall \alpha \in R)$

$$= \prod_{\alpha \in R_+} |1 - e^{-\alpha(x)}|^2$$

□

(30.2) Let $D(t) = |\Delta(x)|^2$ $(t = \exp(x) \in T)$

i.e. $$D(t) = \prod_{\alpha \in R_+} |1 - \xi_{-\alpha}(t)|^2 .$$

Lemma. There exist normalized left-invariant measures $\mu_G, \mu_T, \mu_{G/T}$ on G, T & G/T respectively so that

$$\int_G f(g)\, d\mu_G = \int_{G/T} \left(\int_T f(gt)\, d\mu_T \right) d\mu_{G/T}$$

We will prove this lemma later.

(30.3) Weyl Integration formula

$$\int_G F(g)\, d\mu_G(g) = \frac{1}{|W|} \int_T \left[\int_{G/T} F(g t \bar{g}^{-1})\, d\mu_{G/T}(\bar{g}) \right] D(t)\, d\mu_T(t)$$

(30.4) Special case: $F(gxg^{-1}) = F(x)$ (such functions on G are often called <u>class functions</u>).

$$\int_G F(g)\, d\mu_G(g) = \frac{1}{|W|} \int_T F(t)\, D(t)\, d\mu_T(t)$$

For now, we assume the validity of these formulae and proceed with a proof of Weyl Character formula.

(30.5) We have the following analogue of well known (& easy to prove) identity
$$\frac{1}{2\pi} \int_{S^1} e^{ik\theta}\, d\theta = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

Lemma: Let $\xi_1, \xi_2 : T \to S^1$ be multiplicative characters

Then
$$\int_T \xi_1(t) \overline{\xi_2(t)}\, d\mu_T(t) = \begin{cases} 1 & \text{if } \xi_1 = \xi_2 \\ 0 & \text{o/w} \end{cases}$$

(30.6) For $\lambda \in P_+^{an}$ (i.e. $\lambda \in \mathfrak{h}^*$ s.t. $\lambda(h_i) \geq 0$, $\lambda(h_i) \in \mathbb{Z}$ and λ comes from a character $\xi_\lambda : T \to S^1$)., define

$$\Theta_\lambda(x) = \frac{\sum_{w \in W} \varepsilon(w)\, e^{(w(\lambda+\delta))(x)}}{\Delta(x)} \qquad (x \in \mathfrak{t}_0)$$

Lemma. Θ_λ is W-invariant and descends to a W-inv. function (which we also denote by Θ_λ) on T.

Proof. Let $i \in \{1, \dots, \ell\}$. For the numerator of Θ_λ:

$$\text{Numerator}(s_i x) = \sum_{w \in W} \varepsilon(w)\, e^{(s_i w(\lambda+\delta))(x)} \qquad \left\{ \begin{array}{l} v = s_i w \\ \varepsilon(v) = -\varepsilon(w) \end{array} \right.$$

$$= -\sum_{v \in W} \varepsilon(v)\, e^{(v(\lambda+\delta))(x)} = -\text{Numerator}(x)$$

$$\Delta(s_i x) = \prod_{\alpha \in R_+} e^{\frac{s_i(\alpha)(x)}{2}} - e^{-\frac{s_i(\alpha)(x)}{2}}$$

$$= \left[\prod_{\substack{\alpha \in R_+ \\ \alpha \neq \alpha_i}} \left(e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}} \right) \right] \cdot \left(e^{-\frac{\alpha_i(x)}{2}} - e^{\frac{\alpha_i(x)}{2}} \right)$$

$$= -\Delta(x).$$

Hence $\Theta_\lambda(x)$ is W-invariant.

Rewrite
$$\Theta_\lambda(x) = \frac{\sum_{w\in W} \varepsilon(w)\, e^{(w(\lambda+\delta)-\delta)(x)}}{\prod_{\alpha\in R_+} 1 - e^{-\alpha(x)}}$$

Since $x \mapsto e^{\lambda(x)}$ descends to $\xi_\lambda : T \to S^1$ by assumption,

so does $x \mapsto e^{(w\lambda)(x)}$ since W acts on T $\left(\xi_{w\lambda}(t) = \xi_\lambda(w^{-1}t) \right)$.

$\forall\, w \in W$

It remains to show that $x \longmapsto e^{(w(\delta)-\delta)(x)}$ descends.

By induction on length of w. $\ell(w) = 0 \Rightarrow w = \text{identity}$ ✓.

If $w = s_i u$ and the assertion has been proved for u, we get

$$e^{(w(\delta)-\delta)(x)} = e^{(u(\delta) - s_i\delta)(s_i\cdot x)} = e^{(u(\delta)-\delta+\alpha_i)(s_i\cdot x)}$$
$$= e^{-\alpha_i(x)} \cdot e^{(u(\delta)-\delta)(s_i\cdot x)} \qquad \square$$

(30.7) By Lemma (27.2) page 2, we may regard Θ_λ as a class function on G. Formula (30.4) applies and can be used to prove

Theorem.
$$\int_G \Theta_\lambda(g)\, \overline{\Theta_\mu(g)}\, d\mu_G = \delta_{\lambda\mu}$$

i.e. $\{\Theta_\lambda\}_{\lambda\in P_+^{an}}$ form an orthonormal set in $L^2(G)_{\text{class}} \subset L^2(G)$.

Proof. Formula (30.4) implies that the L.H.S. equals

$$\frac{1}{|W|}\sum_{w_1, w_2} \mathcal{E}(w_1 w_2) \int_T e^{(w_1(\lambda+\delta) - w_2(\mu+\delta))(x)}$$

Now both $\lambda+\delta$ and $\mu+\delta$ take strictly positive values on each h_i. Using Problem 3 of HW4, $w_1(\lambda+\delta) = w_2(\mu+\delta) \iff \begin{matrix}\lambda=\mu\\ w_1=w_2\end{matrix}$

Lemma (30.5) $\Rightarrow$ L.H.S. $= \begin{matrix} 1 & \text{if } \lambda=\mu \\ 0 & \text{if } \lambda\neq\mu \end{matrix}$ $\square$

(30.8) Recall the assertion of Peter-Weyl Theorem (Section (24.4) page 3). Let V_μ $(\mu \in \Lambda_G)$ be the set of iso classes of irred. f.d. repns of G. $(\cdot,\cdot)_{V_\mu}$ G-invariant Hermitian form and $\{v_1(\mu) \dots v_{d_\mu}(\mu)\}$ o.n. basis of V_μ. By Schur's orthogonality relations (section 21.6, item (d) page 9), the matrix coefficients

$\left\{ d_\mu^{1/2}\, \Phi^{V_\mu}_{v_i(\mu), v_j(\mu)} \right\}$ form an o.n. set in $L^2(G)$, which is a basis by Peter-Weyl Theorem. (recall $\Phi^V_{v,w}(g) := (gv, w)$)

$$L^2(G)_{class} := \left\{ f \in L^2(G) : f(gxg^{-1}) = f(x) \right\}$$

$\forall \mu \in \Lambda_G$, let $\chi_\mu = \chi_{V_\mu} : g \longmapsto \mathrm{Tr}(g \text{ acting on } V_\mu)$.

Thm. $\{\chi_\mu\}_{\mu \in \Lambda_G}$ form an o.n. basis of $L^2(G)_{\text{class}}$.

Proof. $\chi_\mu(g) = \sum_{i=1}^{d_\mu} \Phi^{V_\mu}_{v_i(\mu), v_i(\mu)}(g) \Rightarrow$ Orthonormality.

Let $f \in L^2(G)_{\text{class}} \subset L^2(G)$. By Peter-Weyl

$$f = \sum_{\substack{\mu \in \Lambda_G \\ i,j \in \{1,\dots,d_\mu\}}} c_{\mu; i,j} \, \Phi^\mu_{i,j} \qquad \begin{cases} c_{\mu;i,j} \in \mathbb{C} \\ \sum |c_{\mu;i,j}|^2 = \|f\|^2_{L^2} < \infty \end{cases}$$

Now $\forall h \in G$, $\displaystyle f(h) = \int_G f(ghg^{-1}) \, d\mu(g)$.

$$= \sum c_{\mu;i,j} \left(ghg^{-1} v_i , v_j \right)_{V_\mu}$$

<u>Lemma</u> For V a f.d. irred. rep. of G, $X : V \to V$ $\mathbb{C}$-linear

$\displaystyle \overline{X} := \int_G g X g^{-1} \, d\mu(g) : V \to V$ is G-linear, hence

a scalar $c = \dfrac{\mathrm{Tr}(X)}{\dim V}$. (see Lecture 21, pages 9-10 for a proof).

Lemma $\Rightarrow$ $\displaystyle f(h) = \sum c_{\mu;i,i} \frac{\mathrm{Tr}(h \text{ acting on } V_\mu)}{\dim V_\mu}$

$$= \sum_{\mu \in \Lambda_G} \left(\frac{\sum_i c_{\mu;i,i}}{\dim V_\mu} \right) \chi_\mu(h)$$

$\square$