Lecture 31

(31.0) Recall: G - cpct connected Lie group $T \subset G$ max'l torus.

$$\theta_\lambda(t) := \frac{\sum_{w \in W} \varepsilon(w)\, \xi_{w(\lambda+\delta)-\delta}(t)}{\prod_{\alpha \in R_+} 1 - \xi_{-\alpha}(t)} \in L^2(G)_{class}$$

$(\lambda \in P_+^{an})$

$$\chi_\mu(g) = \mathrm{Tr}(g \text{ acting on } V_\mu) \in L^2(G)_{class}$$

$(\mu \in \Lambda_G)$

- $\{\chi_\mu\}_{\mu \in \Lambda_G}$ form an orthonormal basis of $L^2(G)_{class}$
- $\{\theta_\lambda\}_{\lambda \in P_+^{an}}$ form an orthonormal set in $L^2(G)_{class}$

(31.1) Theorem $\Lambda_G = P_+^{an}$, $\chi_\lambda = \theta_\lambda \quad \forall \lambda \in \Lambda_G = P_+^{an}$.

Proof. Let $\mu \in \Lambda_G$ and V_μ corresponding f.d. irred. repn.

(Weight space decomposition) $\forall\, \xi : T \to S^1$

$$V_\mu[\xi] := \{v \in V_\mu : t.v = \xi(t)v \ \forall t \in T\}$$

$$V_\mu = \bigoplus V_\mu[\xi] \quad \text{and} \quad \chi_\mu(t) = \sum_\xi \dim(V[\xi]) \cdot \xi(t)$$

We use additive notation $\xi(t) = e^{\nu(x)}$ $\quad t = \exp(x) \quad x \in \mathfrak{t}_0$

$\nu \in P^{an} \rightsquigarrow \nu : \mathfrak{t}_0 \to i\mathbb{R}$ real linear.

Now χ_μ is W-invariant function. Thus $\Delta \cdot \chi_\mu$ is W-skew-invariant ($\Delta(wx) = \varepsilon(w)\,\Delta(x)$ $\forall\, x \in \mathfrak{t}_0$). (see (30.6) page 4)

i.e. $\forall\, x \in \mathfrak{t}_0$

$$\chi_\mu(\exp(x)).\,\Delta(x) = e^{\delta(x)} \cdot \prod_{\alpha \in R_+} 1 - e^{-\alpha(x)} \cdot \sum_{\nu \in P^{an}}' \dim V_\mu[\nu]\, e^{\nu(x)}$$

$$= \sum_{\nu_j \in P^{an}}' m_j\, e^{(\nu_j+\delta)(x)} \qquad (m_j \in \mathbb{Z})$$

By W-skew-symmetry, and the fact that W-acts simply-transitively on the set of chambers, closure of fundamental chamber is the fundamental domain for W-action ((19.1) page 2 ; HW4, problem 3)

we can choose $\nu_1+\delta, \dots, \nu_k+\delta$ s.t. $\nu_j+\delta\,(h_{j'}) \in \mathbb{Z}_{\geq 0}$

$$\chi_\mu(\exp(x)).\,\Delta(x) = \sum_{j=1,\dots,k} m_j \cdot \left[\sum_{w \in W} \varepsilon(w)\, e^{w(\nu_j+\delta)\,(x)} \right].$$

Note if $\nu_j+\delta\,(h_{j'})=0$ for some j' then

$$\sum \varepsilon(w)\, e^{w(\nu_j+\delta)} = \sum \varepsilon(w)\, e^{w s_{j'}(\nu_j+\delta)} = -\sum_{v \in W} \varepsilon(v)\, e^{v\cdot(\nu_j+\delta)}$$

$\Rightarrow$ it is zero.

Thus we can assume $\nu_j+\delta\,(h_{j'}) \in \mathbb{Z}_{>0}$ $\forall\, j=1,\dots,k$

i.e. each $\nu_j \in P_+^{an}$ $(j=1\dots k)$.

$$\Rightarrow \Delta\cdot\chi_\mu = \sum_{j=1}^{k} m_j\, \Delta\cdot\Theta_{\nu_j} \qquad (m_j \in \mathbb{Z})$$

$$\Rightarrow \chi_\mu = \sum_{j=1}^{k} m_j\, \Theta_{\nu_j}.$$ But $\|\chi_\mu\|^2 = 1$ and $\{\Theta_{\nu_j}\}_{j=1}^{k}$ are o.n.

$$\Rightarrow \sum_{j=1}^{k} |m_j|^2 = 1 \Rightarrow \exists\, j_0 \; m_{j_0} = \pm 1,\ m_j = 0 \text{ for } j \neq j_0 \text{ i.e.}$$

$\chi_\mu = \pm\Theta_{\nu_{j_0}}$. By comparing the leading terms, we get $\nu_{j_0} = \mu$ and hence $\chi_\mu = \Theta_\mu$ and $\mu \in P_+^{an}$.

Now if $\exists\, \lambda \in P_+^{an} \setminus \Lambda_G$, then $\Theta_\lambda \perp \underset{\chi_\mu}{\underset{\|}{\Theta_\mu}}\ \forall \mu \in \Lambda_G$

but $\{\chi_\mu\}_{\mu\in\Lambda_G}$ form an o.n. basis of $L^2(G)_{class}$ - a contradiction. $\square$

(31.2) Cor. (1) $$\prod_{\alpha\in R_+} 1 - e^{-\alpha} = \sum_{w\in W} \varepsilon(w)\, e^{w(\delta)-\delta}$$

(from the character of the trivial repn.)

(2) $$\dim V_\lambda = \frac{\prod_{\alpha\in R_+} (\lambda+\delta, \alpha)}{\prod_{\alpha\in R_+} (\delta, \alpha)}$$

Proof of (2). $\dim V_\lambda = \chi_\lambda(e)$. Both numerator and denominator of the Weyl character formula vanish at $t = e$

with order of vanishing = # of positive roots. So, we need to use L'Hospital rule.

$\forall \alpha \in R_+$ define $\partial_\alpha \circlearrowright$ Functions on $\mathfrak{h}$ as directional derivative in the direction of t_α, i.e.

$$(\partial_\alpha f)(h) = \lim_{\varepsilon \to 0} \frac{f(h+\varepsilon t_\alpha) - f(h)}{\varepsilon}$$

Then $\partial_\alpha e^\beta = (\beta, \alpha) e^\beta$. Now we apply $\prod_{\alpha \in R_+} \partial_\alpha$:

$$\prod_{\alpha \in R_+} \partial_\alpha \cdot \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\delta)} = \sum_{w \in W} \varepsilon(w) \left[\prod_{\alpha \in R_+} (\lambda+\delta, w^{-1}\alpha) \right] \cdot e^{w(\lambda+\delta)}$$

<u>Claim</u>: $|w^{-1} R_+ \cap R_-| = \ell(w^{-1})$

Claim $\Rightarrow$ $\varepsilon(w). \prod_{\alpha \in R_+} (\lambda+\delta, w^{-1}\alpha) = \prod_{\alpha \in R_+} (\lambda+\delta, \alpha)$

Same calculation with $(\lambda \mapsto 0)$ and evaluating at $h = 0$ gives Weyl dimension formula.

<u>Proof of the claim</u>: Let $S(w) = w R_+ \cap R_-$.

$S(e) = \phi$, hence $|S(e)| = \ell(e)$

$S(s_i) = \{-\alpha_i\}$, hence $|S(s_i)| = 1 = \ell(s_i)$

If $w \in W$ is of length k and $w = u s_i$, $\ell(u) = k-1$

then by Prop 19.4, page 3, we get

$$\delta(w) = \{-\alpha_i\} \cup \delta(u)$$ and we are done by

induction $\square$

(31.3) Examples. I. Type A_2 $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \leftarrow \alpha_i(h_j)$

$$\omega_1 = \frac{2\alpha_1 + \alpha_2}{3} \qquad \delta = \omega_1 + \omega_2 = \alpha_1 + \alpha_2$$

$$\omega_2 = \frac{\alpha_1 + 2\alpha_2}{3} \qquad R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$$

$$\prod_{\alpha \in R_+} (\delta, \alpha) = 1.1.2 = 2$$

$\Rightarrow \dim V_{\omega_1} = 3$

$\lambda = \omega_1$: $\prod (\lambda + \delta, \alpha) = 2.1.3 = 6$

standard repn. of $SU(3)$