Lecture 32

(32.0) Recall: we proved the following results for a compact Lie group G. (1) $\Lambda_G = P_+^{an}$ (set of iso. classes of f.d. irred. repns. of G is in bijection with the set of (analytic) dominant weights) (2) $\forall \lambda \in P_+^{an}$, $\chi_\lambda = \chi_{V_\lambda}$ (character of V_λ)

$$\chi_\lambda = \frac{\sum_{w \in W} \varepsilon(w)\, \xi_{w(\lambda+\delta)-\delta}}{\prod_{\alpha \in R_+} (1 - \xi_{-\alpha})}$$

$$\left(\begin{array}{l} \text{here } W = \text{Weyl gp} = N_G(T)/T \\ R_+ = \text{set of positive roots} \\ \delta = \frac{1}{2} \sum_{\alpha \in R_+} \alpha \end{array} \right)$$

Cor. (1) Weyl denominator formula: $\prod_{\alpha \in R_+} 1 - \xi_{-\alpha} = \sum_{w \in W} \varepsilon(w)\, \xi_{w(\delta)-\delta}$

(2) Weyl dimension formula: $\dim V_\lambda = \prod_{\alpha \in R_+} \frac{(\lambda+\delta, \alpha)}{(\delta, \alpha)}$

(32.1) Example A_2 $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. $W = S_3 = \{1, s_1, s_2, s_1 s_2, s_2 s_1, \underset{\substack{\| \\ s_2 s_1 s_2}}{s_1 s_2 s_1}\}$

$R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

$\omega_1 = \frac{2\alpha_1 + \alpha_2}{3}$ $\quad \omega_2 = \frac{\alpha_1 + 2\alpha_2}{3}$ $\quad \delta = \omega_1 + \omega_2 = \alpha_1 + \alpha_2$

$s_i \omega_j = \omega_j - \omega_j(h_i)\alpha_i = \omega_j - \delta_{ij}\alpha_i$

$s_1(\alpha_2) = \alpha_2 + \alpha_1 = s_2(\alpha_1)$ and $s_i(\alpha_i) = -\alpha_i$

	1	s_1	s_2	$s_1 s_2$	$s_2 s_1$	$s_1 s_2 s_1 = s_2 s_1 s_2$
ω_1	ω_1	$\omega_1-\alpha_1$	ω_1	$\omega_1-\alpha_1$	$\omega_1-\alpha_1-\alpha_2$	$\omega_1-\alpha_1-\alpha_2$
ω_2	ω_2	ω_2	$\omega_2-\alpha_2$	$\omega_2-\alpha_1-\alpha_2$	$\omega_2-\alpha_2$	$\omega_2-\alpha_1-\alpha_2$
$\delta=\omega_1+\omega_2$	δ	$\delta-\alpha_1$	$\delta-\alpha_2$	$\delta-2\alpha_1-\alpha_2$	$\delta-\alpha_1-2\alpha_2$	$\delta-2\alpha_1-2\alpha_2$

$$\text{Numerator of } \chi_{\omega_1} = \sum_{w\in W} \varepsilon(w)\, \xi_{w(\omega_1+\delta)-\delta}$$

$$= \xi_{-\delta}\left\{ \xi_{\omega_1+\delta}\left(1-\xi_{-2\alpha_1}-\xi_{-\alpha_2}+\xi_{-3\alpha_1-\alpha_2}+\xi_{-2\alpha_1-3\alpha_2}-\xi_{-3\alpha_1-3\alpha_2}\right)\right\}$$

$$\text{Denominator of } \chi_{\omega_1} = \left(1-\xi_{-\alpha_1}\right)\left(1-\xi_{-\alpha_2}\right)\left(1-\xi_{-\alpha_1-\alpha_2}\right)$$

For ease of calculation set $z_1 = \xi_{-\alpha_1}$ $\quad z_2 = \xi_{-\alpha_2}$

$$\text{So } \chi_{\omega_1} = \xi_{\omega_1}\left\{\frac{1-z_1^2-z_2+z_1^3z_2+z_1^2z_2^3-z_1^3z_2^3}{(1-z_1)(1-z_2)(1-z_1z_2)}\right\}$$

$$\text{Now } 1-z_1^2-z_2+z_1^3z_2+z_1^2z_2^3-z_1^3z_2^3 = (1-z_1)\left(1+z_1-z_2-z_1z_2-z_1^2z_2+z_1^2z_2^3\right)$$

$$= (1-z_1)(1-z_2)\left(1+z_1-z_1^2z_2-z_1^2z_2^2\right)$$

$$= (1-z_1)(1-z_2)(1-z_1z_2)(1+z_1+z_1z_2)$$

$$\text{So } \chi_{\omega_1} = \xi_{\omega_1}\left(1+\xi_{-\alpha_1}+\xi_{-\alpha_1-\alpha_2}\right)$$

(32.2) Proof of Weyl integration formula.

I. G - invariant (left) integration on G/T.

$\forall\ f \in C(G)$ (= cnts. fns. $G \longrightarrow \mathbb{R}$), define

$f_* \in C(G/T)$ by $\quad f_*(\bar{g}) = \displaystyle\int_T f(gt)\, d\mu_T(t)$

Note: μ_T is normalized so that $\mu_T(T) = 1$. This implies, if $pr: G \longrightarrow G/T$ is the projection, then $\forall\ h \in C(G/T)$

$$(h \circ pr)_* = h$$

Thus we can define $\displaystyle\int_{G/T}$ so that $\forall\ f \in C(G)$

$$\int_{G/T} f_*(\bar{g})\, d\mu_{G/T}(\bar{g}) = \int_G f(g)\, d\mu_G(g)$$

same as $\displaystyle\int_{G/T} \left(\int_T f(gt)\, d\mu_T(t) \right) d\mu_{G/T}(\bar{g})$

That is, for any function $h \in C(G/T)$, choose $f \in C(G)$

so that $f_* = h$ and define

$$\int_{G/T} h(\bar{g})\, d\mu_{G/T}(\bar{g}) := \int_G f(g)\, d\mu(g)$$

Ex. This definition is independent of the choice of f.

II. Now consider $\psi : G/T \times T \longrightarrow G$

$$(g, t) \longmapsto g t g^{-1}$$

From HW 6, problem 3

$$T\psi \text{ (at } (g,t)) : (X, Y) \longmapsto Ad(g)\left((Ad(t^{-1}) - 1)X + Y\right)$$

$$X \in T_{\bar{g}}(G/T) \simeq \mathfrak{t}_0^{\perp} \subset \mathfrak{g}_0$$

$$Y \in T_t(\text{Max'l torus}) \simeq \mathfrak{t}_0$$

Cor. Determinant of differential of ψ at $(\bar{g}, t)$ is given by $\prod_{\alpha \in R} 1 - \xi_{-\alpha}(t)$

III. $T^{reg} := \{ t \in T \text{ s.t. } Z_G(t)^0 = T \}$

Ex. Let $t \in T$. Then $t \in T^{reg} \iff \prod_{\alpha \in R} 1 - \xi_{-\alpha}(t) \neq 0$

Consider $\psi : G/T \times T^{reg} \longrightarrow G^{reg}$

Ex. $\cancel{\psi \times X \times}$ $\forall\, g \in G^{reg}$, $|\psi^{-1}(g)| = |W|$.

Ex. If $\phi : M \longrightarrow N$ is orientation preserving, everywhere regular (i.e. $T_m\phi$ is non-singular $\forall\, m \in M$) smooth map s.t. $\forall\, q \in N$, $|\phi^{-1}(q)| = n$. Then

$$n \cdot \int_N f\omega = \int_M (f \circ \phi)(\phi^{*}\omega)$$

Finally it remains to observe that $T \setminus T^{reg}$ is of lower dim hence of measure 0. Combining all these observations we get

$$\int_G f(g)\, d\mu_G(g) = \frac{1}{|W|} \int_T \left[\int_{G/T} f(g t \bar{g}^{-1})\, d\mu_{G/T}(\bar{g}) \right] \cdot D(t)\, d\mu_T(t)$$

and the Weyl integration formula follows.