Lecture 33

(33.0) Aim: to construct irred. f.d. repns. of a (semi-) simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Recall the notations

$\mathfrak{g}$: simple Lie algebra over $\mathbb{C}$. $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra

$R \subset \mathfrak{h}^* \setminus \{0\}$ set of roots $\{\alpha_1, \dots, \alpha_\ell\} \subset R$ a base of R.

W = Weyl group $A = (a_{ij})_{1 \le i,j \le \ell}$ Cartan matrix of $\mathfrak{g}$

$$\begin{array}{ccc} \mathfrak{h}^* & \xrightarrow{\sim} & \mathfrak{h} \\ \psi & & \psi \\ \alpha & \longmapsto & t_\alpha \end{array} \qquad h_\alpha := \frac{2 t_\alpha}{|\alpha|^2} \qquad a_{ij} := \alpha_j(h_i)$$

defined using non-deg form:
$\alpha(x) = (t_\alpha, x) \ \forall x \in \mathfrak{h}$

root lattice; weight lattice

Lattices $$Q = \sum_{\alpha \in R} \mathbb{Z}\alpha \qquad P = \{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \in \mathbb{Z} \ \forall \alpha \in R\}$$

$$\cup \qquad\qquad\qquad \cup$$

$$Q_+ = \sum_{\alpha \in R_+} \mathbb{N}\alpha \qquad P_+ = \{\lambda \in P : \lambda(h_i) \ge 0 \ \forall i = 1, \dots, \ell\}$$

dominant weights

Partial order on $\mathfrak{h}^*$: $\lambda \ge \mu \iff \lambda - \mu \in Q_+$.

(33.1) Recall that we proved complete reducibility of f.d. repns. of $\mathfrak{g}$ (Thm. (9.6) page 10 of Lecture 9)

Let V be a f.d. irred. repn. of $\mathfrak{g}$.

Prop. (1) $\mathfrak{h}$ acts diagonally on V. That is, $\forall \mu \in \mathfrak{h}^*$ define

$$V[\mu] = \{v \in V : h \cdot v = \mu(h) v\} \ (=: \mu\text{-weight space of } V)$$

Then $$V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu] \quad (\text{as } \mathfrak{h}\text{-repn.})$$

(2) Let $P(V) := \{\mu \in \mathfrak{h}^* : V[\mu] \neq 0\}$ (finite set since $\dim V < \infty$)

Then $\exists! \ \lambda \in P(V)$ s.t.

$$\dim V[\lambda] = 1 \quad \& \quad \mu \in P(V) \Rightarrow \mu \leq \lambda$$

Moreover, $\lambda \in P_+$ (and hence $P(V) \subset P$)

Proof. (1) Let $V' = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu]$. Since $\mathfrak{h} \circlearrowright V$ via commuting matrices (as $\mathfrak{h}$ is abelian), there exists a joint eigenvector for $\mathfrak{h}$. $\Rightarrow V' \neq (0)$. Moreover by the decomposition of $\mathfrak{g}$ into root spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

, we know that $\forall \ x \in \mathfrak{g}_\alpha$, $\mu \in \mathfrak{h}^*$ if $v \in V[\mu]$ (assuming $\neq 0$) then

x maps $V[\mu]$ to $V[\mu+\alpha]$:

$$\begin{aligned} h.(x.v) &= [h,x].v + x.(h.v) \\ &= \alpha(h)\, x.v + \mu(h)\,(x.v) \\ &= (\mu+\alpha)(h)\,(x.v) \quad \forall h \in \mathfrak{h}. \end{aligned}$$

$\Rightarrow V' \subset V$ is a non-zero subrepn. By irreducibility of V,

$V = V'$.

(2) Since $P(V)$ is a finite set, let $\lambda \in P(V)$ be maximal w.r.t. $\leq$ on $\mathfrak{h}^*$. Then for each $\alpha \in R_+$, $e_\alpha \in \mathfrak{g}_\alpha$ basis vector

1-dim'l

$$e_\alpha V[\lambda] \subset V[\lambda+\alpha] = 0$$ by maximality of λ.

As $V[\lambda] \neq 0$, let $v \in V[\lambda]$ be a non-zero vector. Then

$$h\cdot v = \lambda(h) v \quad \text{and} \quad e_\alpha v = 0 \quad \forall \alpha \in R_+ .$$

Let $V'' \subset V$ be the subrepn. generated by v. Then

$$V''[\lambda] = \mathbb{C}\cdot v \quad \text{1-dim'l} \qquad \mathfrak{g} = \bigoplus_{\alpha\in R_+} \mathfrak{g}_{-\alpha} \oplus \underbrace{\mathfrak{h}}_{\substack{\text{acts by}\\ \lambda\in\mathfrak{h}^* \text{ on } v}} \oplus \underbrace{\bigoplus_{\alpha\in R_+} \mathfrak{g}_\alpha}_{\text{acts as 0 on } v}$$

$$V''[\mu] \neq 0 \Rightarrow \mu \leq \lambda$$

but by irreducibility of V, $V'' = V \Rightarrow$ same is true for V.

$\underline{\lambda \in P_+}$: Recall the $\mathfrak{sl}_2$-calculation: $V \circlearrowleft \mathfrak{sl}_2$ ($\cancel{\text{f.d.}}$) a repn

$0 \neq v \in V$ s.t. $\begin{array}{l} ev = 0 \\ hv = \mu v \ (\mu\in\mathbb{C}) \end{array}$. Then $\forall k \geq 0$

$$e\cdot f^k v = k(\mu - k + 1) f^{k-1} v$$

Now $\forall i \in \{1, \dots, \ell\}$, $v\in V[\lambda]$ chosen non-zero vector, we have

$$\begin{array}{l} e_i v = 0 \\ h_i v = \lambda(h_i) v \end{array} \Rightarrow e_i(f_i^k v) = k(\lambda(h_i) - k + 1) f_i^{k-1} v$$

As $P(V)$ is finite and $f_i^k v \in V[\lambda - k\alpha_i]$, for k large enough $f_i^k = 0$. Let $n_i \in \mathbb{N}$ be the smallest s.t. $f_i^{n_i} v \neq 0$ but $f_i^{n_i+1} v = 0$

$$\text{Then} \quad 0 = e_i\cdot(f_i^{n_i+1} v) = (n_i+1)(\lambda(h_i) - n_i) \underbrace{f_i^{n_i} v}_{\neq 0}$$

$$\Rightarrow \lambda(h_i) = n_i \in \mathbb{Z}_{\geq 0}$$

$\square$

(33.2) As a consequence of Prop. (33.1) we have a well-defined map

$$\mathrm{Irr}(\mathfrak{g}) \longrightarrow P_+ \qquad (V \longmapsto \text{highest weight of } V).$$

(set of iso. classes of f.d. irred. repns. of $\mathfrak{g}$)

To prove this map is a bijection, we need the construction of Verma modules.

(33.3) Enveloping algebra of $\mathfrak{g}$, $\mathcal{U}(\mathfrak{g})$, is defined as

(1) $$\mathcal{U}(\mathfrak{g}) := \frac{\text{Tensor algebra of } \mathfrak{g} \ (= \text{free unital assoc. algebra over } \mathfrak{g})}{\langle x\otimes y - y\otimes x - [x,y] \ : \ x,y\in\mathfrak{g}\rangle \ \text{(2-sided ideal)}}$$

(Tensor algebra of a f.d. $\mathbb{C}$-vector space (say F) $= \mathbb{C}\oplus F\oplus(F\otimes F)\oplus\cdots\oplus(F^{\otimes n})\oplus\cdots$

$=$ algebra of non-commutative polynomials in $\dim F$ variables)

(2) $\mathfrak{g} \xhookrightarrow{i} \mathcal{U}(\mathfrak{g})$ and $i[X,Y] = i(X)\,i(Y) - i(Y)\,i(X)$

(3) $\mathcal{U}(\mathfrak{g})$ is universal w.r.t (2). That is, if A is any unital assoc. alg. together with a $\mathbb{C}$-linear map $\alpha : \mathfrak{g}\to A$ s.t.

$$\alpha([X,Y]) = \alpha(X)\,\alpha(Y) - \alpha(Y)\,\alpha(X) \qquad \forall\, X,Y\in\mathfrak{g}$$

Then $\exists!$ alg. hom. $\mathcal{U}(\mathfrak{g}) \xrightarrow{\beta} A$ s.t. $\beta\circ i = \alpha$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\alpha"] \arrow[d, "i"'] & A \\ \mathcal{U}(\mathfrak{g}) \arrow[ur, dashed, "\exists!\,\beta"'] & \end{tikzcd}$$

(universal property of $\mathcal{U}(\mathfrak{g})$)

Theorem (Poincaré - Birkhoff - Witt) (cf Jurjei Chen's talk)

Let $\{x_i\}_{i=1\ldots N}$ be a basis of $\mathfrak{g}$ (ordered in some way). Then the set of ordered monomials $\{x_{i_1} \cdots x_{i_k}\}_{1 \le i_1 \le \ldots \le i_k \le N}$ form a basis of $\mathcal{U}(\mathfrak{g})$.

Examples (i) $\mathfrak{g} = \mathfrak{sl}_2$ (basis $\{f, h, e\}$, $[h\,e] = 2e$, $[h\,f] = -2f$, $[e, f] = h$)

$$\mathcal{U}(\mathfrak{sl}_2) = \mathbb{C}\langle F, H, E \rangle \Big/ \left\langle \begin{array}{l} HE - EH = 2E \\ HF - FH = -2F \\ EF - FE = H \end{array} \right\rangle$$

basis $\{F^a H^b E^c : a, b, c \in \mathbb{Z}_{\ge 0}\}$

(ii) $\mathfrak{g} = \mathfrak{sl}_3$. Basis h_1, h_2 ; $e_1, e_2, e_3 = [e_1, e_2]$; $f_1, f_2, f_3 = [f_1, f_2]$

$$\mathcal{U}(\mathfrak{sl}_3) = \mathbb{C}\langle h_1, h_2, e_1, e_2, f_1, f_2 \rangle \Big/ \begin{array}{l} h_1 h_2 = h_2 h_1 \\ h_i e_j = e_j h_i + a_{ij} e_j \\ h_i f_j = f_j h_i - a_{ij} f_j \\ e_i f_j = f_j e_i + \delta_{ij} h_i \\ (1 \le i, j \le 2) \end{array}$$

Serre Rel^{n}s $(i \ne j)$

$$f_i^2 f_j - 2 f_i f_j f_i + f_j f_i^2 = 0$$

$$e_i^2 e_j - 2 e_i e_j e_i + e_j e_i^2 = 0$$

PBW basis $\left\{ f_3^{n_3} f_1^{n_1} f_2^{n_2} h_1^{k_1} h_2^{k_2} e_2^{m_2} e_1^{m_1} e_3^{m_3} : \begin{array}{c} n_1, n_2, n_3 \\ m_1, m_2, m_3 \in \mathbb{Z}_{\ge 0} \\ k_1, k_2 \end{array} \right\}$

Uses one more straightening relation

$$f_1 f_2 = f_2 f_1 + f_3.$$

$$e_1 e_2 = e_2 e_1 + e_3$$

(33.4) Recall : simple Lie alg. $\mathfrak{g}$ associated to $A = (a_{ij})_{1 \le i,j \le \ell}$ has the following presentation on generators $\{h_i, e_i, f_i\}_{1 \le i \le \ell}$

$$\forall i,j \in \{1,\dots,\ell\} \quad \begin{cases} [h_i, h_j] = 0 \qquad [h_i, e_j] = a_{ij} \cdot e_j \qquad [h_i, f_j] = -a_{ij} f_j \\ [e_i, f_j] = \delta_{ij} h_i \end{cases}$$

For $i \neq j$, let $m = 1 - a_{ij}$. Then $\mathrm{ad}(e_i)^m e_j = 0 = \mathrm{ad}(f_i)^m f_j$

In $\mathcal{U}(\mathfrak{g})$ $\quad \mathrm{ad}(e_i)^m e_j = \displaystyle\sum_{s=0}^{m} (-1)^s \binom{m}{s} e_i^{m-s} e_j e_i^s$

$\mathcal{U}(\mathfrak{g})$ = free alg. (unital assoc.) over 3ℓ variables h_i, e_i, f_i $(i = 1, \dots, \ell)$ subject to rel^{n}s listed above.

<u>Basis of $\mathfrak{g}$</u> : $\quad \mathfrak{g} = \left(\bigoplus_{\alpha \in R_+} \mathfrak{g}_{-\alpha} \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \right)$

$\{f_\alpha\}_{\alpha \in R_+}$ $\qquad h_i \ (1 \le i \le \ell)$ $\qquad \{e_\alpha\}_{\alpha \in R_+}$ as $\dim \mathfrak{g}_\alpha = 1$

PBW basis of $\mathcal{U}(\mathfrak{g})$: enumerate $R_+ = \{\alpha^{(1)}, \dots, \alpha^{(N)}\}$; $N = |R_+|$

Ordered products $\quad f_{\alpha^{(1)}}^{n_1} \cdots f_{\alpha^{(N)}}^{n_N} \cdot h_1^{k_1} \cdots h_\ell^{k_\ell} \, e_{\alpha^{(1)}}^{m_1} \cdots e_{\alpha^{(N)}}^{m_N}$

form a basis of $\mathcal{U}(\mathfrak{g})$.