Lecture 34

(34.0) Recall: for a simple Lie algebra $\mathfrak{g}$ (over $\mathbb{C}$), we proved

(1) Every f.d. repn = direct sum of irred. f.d. repns.

(2) V : f.d. irred. $\Rightarrow \exists!\ \lambda \in P_+$ s.t. $\dim V[\lambda] = 1$

$V[\mu] \neq 0 \Rightarrow \mu \leq \lambda$ (highest weight of V)

(34.1) Verma module M_λ $(\lambda \in \mathfrak{h}^*)$ is the universal repn. of $\mathfrak{g}$ containing a highest weight of weight λ.

Defn. $M_\lambda = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b}_+)}{\otimes} \mathbb{C}$ where $\mathfrak{b}_+ = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \subset \mathfrak{g}$ subalg.

$\mathfrak{b}_+ \circlearrowright \mathbb{C}$: $\mathfrak{g}_\alpha$ as 0 $\forall \alpha \in R_+$, $\mathfrak{h}$ via $\lambda : \mathfrak{h} \to \mathbb{C}$

$$= \mathcal{U}(\mathfrak{g}) \Big/ \begin{matrix}\text{Left ideal generated} \\ \text{by } e_\alpha\ (\alpha \in R_+)\ \& \\ h - \lambda(h)\ \forall h \in \mathfrak{h}\end{matrix}$$

PBW $\Rightarrow$ M_λ has a basis consisting of ordered monomials

$\left\{ f_{\alpha^{(1)}}^{n_1} \cdots f_{\alpha^{(N)}}^{n_N} \right\}$ where $\{\alpha^{(1)}, \ldots, \alpha^{(N)}\} = R_+$ (arbitrary enumeration)

Notation $\mathbb{1}_\lambda$ = image of $1 \in \mathcal{U}(\mathfrak{g})$ in M_λ ($\mathcal{U}(\mathfrak{g})$ is unital!)

PBW basis of $M_\lambda = \left\{ f_{\alpha^{(1)}}^{n_1} \cdots f_{\alpha^{(N)}}^{n_N} \cdot \mathbb{1}_\lambda \right\}_{n_1 \ldots n_N \in \mathbb{Z}_{\geq 0}}$

(34.2) Prop. (1) $M_\lambda = \bigoplus_{\mu \le \lambda} M_\lambda[\mu]$ (i.e. $\mathfrak{h}$ acts diagonally

and weights of M_λ, $P(M_\lambda) = \{\lambda - \alpha : \alpha \in Q_+\}$

(2) $M_\lambda[\lambda] = \mathbb{C} \cdot \mathbb{1}_\lambda$ (1-dim'l). More generally

$\dim M_\lambda[\mu]$ = number of ways of writing $\lambda - \mu$ as sum of positive roots

(Kostant's partition function)

(3) Universal Property of M_λ. Let V be a repn. of $\mathfrak{g}$ (not necessarily finite-dim'l or irreducible).

$$\operatorname{Hom}_{\mathfrak{g}\text{-repns}}(M_\lambda, V) = V[\lambda]^{\text{sing}} = \left\{ v \in V : \begin{array}{l} e_\alpha v = 0 \;\; \forall \alpha \in R_+ \\ h.v = \lambda(h) v \end{array} \right\}$$

(4) $\exists!$ proper ~~&~~ max'l submodule $K_\lambda \subset M_\lambda$. The quotient

$L_\lambda = M_\lambda / K_\lambda$ is the unique irreducible repn. of $\mathfrak{g}$ generated by a highest weight vector of highest weight λ

(Hence, $\operatorname{Irr}(\mathfrak{g}) \longrightarrow P_+$ is injective). (see (33.2) page 4).

Proof of (4): If $M' \subset M_\lambda$ is a submodule then $\mathbb{1}_\lambda \in M' \Rightarrow M' = M_\lambda$. So $\mathbb{1}_\lambda \notin M'$ for any proper submodule M' of M_λ. Hence $\sum_{M' \text{ proper submodule}} M' =: K_\lambda$ is the unique max'l proper submod.

If L is an irreducible repn. of $\mathfrak{g}$ s.t. $L[\lambda]$ is ~~f.d.~~ non-zero and $e_\alpha L[\lambda] = 0$ $\forall \alpha \in R_+$, then we get a non-zero morphism

$$\begin{aligned} M_\lambda &\xrightarrow{\ \eta\ } L \\ \mathbb{1}_\lambda &\longmapsto v \in L[\lambda] \quad (v \neq 0) \end{aligned}$$

$\mathrm{Ker}(\eta)$ is a proper submodule $\Longrightarrow$ $\mathrm{Ker}(\eta) \subset K_\lambda$. So η descends to a non-zero injective map $L_\lambda = M_\lambda / K_\lambda \overset{\eta}{\hookrightarrow} L$.

But L is irreducible, hence $L_\lambda \simeq L$. $\square$

(34.3) Category $\mathcal{O}$ (Bernstein-Gelfand-Gelfand) A repn. V of $\mathfrak{g}$ is in category $\mathcal{O}$ if

(1) $V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu]$ $\quad \dim V[\mu] < \infty$ $\quad$ ($\mathfrak{h}$ acts diagonally with f.d. eigenspaces)

(2) $\exists\ \lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ s.t. $V[\mu] \neq 0 \Rightarrow \mu \leq \lambda_j$ for some j

(i.e. $P(V) := \{\mu \in \mathfrak{h}^* : V[\mu] \neq 0\} \subset \bigcup_{j=1}^{r} \lambda_j - Q_+$)

e.g. $M_\lambda \in \mathcal{O}$ $\forall \lambda \in \mathfrak{h}^*$. $\quad \mathrm{Irr}(\mathcal{O}) \longleftrightarrow \mathfrak{h}^*$

e.g. $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda \in \mathbb{C}$, $M_\lambda = \mathrm{Span}$ of $\{f^n \mathbb{1}_\lambda\}_{n \geq 0}$

$$e f^n \mathbb{1}_\lambda = n(\lambda - n + 1) f^{n-1} \mathbb{1}_\lambda$$

so M_λ is irreducible if $\lambda \in \mathbb{C} \setminus \mathbb{N}$. For $\lambda \in \mathbb{N}$ we have a non-split short exact sequence

$$0 \longrightarrow M_{-\lambda-1} \longrightarrow M_\lambda \longrightarrow L_\lambda \longrightarrow 0$$

$(\lambda+1)$-dim'l irred.

(34.4) $M \in \mathcal{O}$ is said to be <u>integrable</u> if $\forall\, m \in M$, $i \in \{1, \dots, \ell\}$, $\exists\, n > 0$ (depending on m and i) s.t. $f_i^n m = 0$. (This defn makes sense w/o $M \in \mathcal{O}$ condition – one imposes e_i's are also locally nilpotent)

Weyl group action. $\forall\, i \in \{1, \dots, \ell\}$ define

$$\tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i)$$

Prop. Let M be an integrable repn. of $\mathfrak{g}$. Then $\tilde{s}_i \in \mathrm{Aut}(M)$ ($\mathbb{C}$-linear automorphism of M)

$$\tilde{s}_i : M[\mu] \xrightarrow{\sim} M[s_i\mu]$$

vector space iso

W-action on $\mathfrak{h}^*$

Proof. Each $\begin{cases} x = e_i \text{ or } f_i \\ m \in M \end{cases}$: $\left(\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot m$ is a finite sum.

So $\tilde{s}_i$ are well-defined operators on M.

Let $m \in M[\mu]$. Let us compute $h \cdot (\tilde{s}_i \cdot m)$ for $h \in \mathfrak{h}$

$$h \cdot (\tilde{s}_i \cdot m) = \tilde{s}_i \left[(\tilde{s}_i^{-1} h\, \tilde{s}_i^{+1})\, m \right]$$

Claim $\tilde{s}_i^{-1} h\, \tilde{s}_i = s_i(h) \quad \forall\, h \in \mathfrak{h}$

$\Rightarrow h \cdot (\tilde{s}_i \cdot m) = \mu(s_i(h))\, (\tilde{s}_i \cdot m)$ and the prop follows

Proof of the claim. $\tilde{s}_i^{-1} h \tilde{s}_i = \exp(\mathrm{ad}(-e_i)) \exp(\mathrm{ad}(f_i)) \exp(-\mathrm{ad}(e_i)) h$

So if $\alpha_i(h) = 0$, we get $\tilde{s}_i^{-1} h \tilde{s}_i = h$.

<u>if $h = h_i$</u> : $\exp(-\mathrm{ad}(e_i)) \cdot h_i = \sum_{n \geq 0} (-1)^n \dfrac{\overbrace{[e_i [e_i \dots [e_i h_i] \dots]]}^{n \text{ times}}}{n!}$

$$\left.\begin{array}{l} [e_i, h_i] = -2 e_i \\ [e_i [e_i h_i]] = 0 \end{array}\right\} \Rightarrow \exp(-\mathrm{ad}(e_i)) \cdot h_i = h_i + 2 e_i$$

$$\exp(\overset{\mathrm{ad}}{} f_i) \exp(-\mathrm{ad}(e_i)) \cdot h_i = \exp(\mathrm{ad}(f_i)) \{ h_i + 2 e_i \}$$

$$= \sum_{n \geq 0} \frac{1}{n!} \Big(\underbrace{[f_i, [f_i, \dots [f_i h_i] \dots]]}_{n \text{ times}} + 2 \underbrace{[f_i [f_i \dots [f_i e_i] \dots]]}_{n\text{-times}} \Big)$$

$$= (h_i + 2 f_i) + 2 (e_i - h_i - f_i)$$

$$\left(\begin{array}{l} [f_i h_i] = +2 f_i \, ; \; [f_i [f_i h_i]] = 0 \\ [f_i, e_i] = -h_i \\ [f_i [f_i e_i]] = -2 f_i \quad \text{then } 0 \end{array} \right)$$

$$= -h_i + 2 e_i$$

Finally $\exp(-\mathrm{ad}(e_i)) \cdot (-h_i + 2 e_i)$

$$= -h_i - 2 e_i + 2 e_i = -h_i$$

□

(34.5) Casimir element. (Lecture 9 page 5)

Lemma. Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{C}$, together with an invariant non-degenerate symmetric bilinear form $(\cdot,\cdot) : \mathfrak{a} \times \mathfrak{a} \to \mathbb{C}$.

Let $\{a_i\}$ be a basis of $\mathfrak{a}$ and $\{a^i\}$ the dual basis (w.r.t. $(\cdot,\cdot)$)

Then $C_{\mathfrak{a}} := \sum_i a_i a^i \in \mathcal{U}(\mathfrak{a})$ satisfies $[X, C_{\mathfrak{a}}] = 0$ $\forall\, X \in \mathfrak{a}$.

Proof. $[X, C_{\mathfrak{a}}] = \sum [X, a_i] a^i + \sum a_i [X, a^i]$. If

$$[X, a_i] = \sum_j c_i^j a_j \qquad [X, a^i] = \sum_j d_j^i a^j$$

then $[X, C_{\mathfrak{a}}] = \sum_{i,j} (c_j^i + d_j^i) a_i a^j$. But

$$c_j^i = ([X, a_j], a^i) = -(a_j, [X, a^i]) = -d_j^i.$$

by invariance

So $[X, C_{\mathfrak{a}}] = 0$ $\square$

Now if $\mathfrak{a} = \mathfrak{g}$ simple Lie algebra over $\mathbb{C}$, $C_{\mathfrak{g}}$ has the following form

$$C_{\mathfrak{g}} = \sum_{i=1}^{\ell} x_i^2 + \sum_{\alpha \in R_+} x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha$$

- $\{x_i\}_{i=1}^{\ell}$ is an o.n. basis of $\mathfrak{h}$
- $\{x_\alpha \in \mathfrak{g}_\alpha\}$ are chosen so that $(x_\alpha, x_{-\alpha}) = 1$ ($\forall\, \alpha \in R_+$).

Using $x_\alpha x_{-\alpha} = x_{-\alpha} x_\alpha + t_\alpha$, we get

$$C_{\mathfrak{g}} = \sum_{i=1}^{\ell} x_i^2 + 2t_\delta + 2\sum_{\alpha\in R_+} x_{-\alpha}x_\alpha$$

Cor. $C_{\mathfrak{g}}$ acts on M_λ by $|\lambda+\delta|^2 - |\delta|^2$

Pf. $C_{\mathfrak{g}}\, \mathbb{1}_\lambda = \left\{ \sum_{i=1}^{\ell} \lambda(x_i)^2 + 2(\lambda,\delta) \right\} \mathbb{1}_\lambda$ since

$h.\mathbb{1}_\lambda = \lambda(h)\, \mathbb{1}_\lambda$ and $x_\alpha \mathbb{1}_\lambda = 0 \quad \forall \alpha\in R_+$.

$\sum_{i=1}^{\ell} \lambda(x_i)^2 = (\lambda,\lambda)$ and we get

$$\begin{aligned} C_{\mathfrak{g}}\, \mathbb{1}_\lambda &= \left\{ (\lambda,\lambda) + 2(\lambda,\delta) + (\delta,\delta) - (\delta,\delta) \right\} \mathbb{1}_\lambda \\ &= \left(|\lambda+\delta|^2 - |\delta|^2 \right) \mathbb{1}_\lambda . \end{aligned}$$

Now M_λ is generated by $\mathbb{1}_\lambda$ and $C_{\mathfrak{g}}$ commutes with elements of $\mathfrak{g}$. Hence $C_{\mathfrak{g}}\big|_{M_\lambda} = \left(|\lambda+\delta|^2 - |\delta|^2 \right) \cdot \mathrm{Id}_{M_\lambda}$ $\square$